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Mathematical modeling on the effects of drug abuse to the societies

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**MATHEMATICAL MODELING ON THE EFFECTS OF DRUG
ABUSE TO THE SOCIETIES**

Fikiri Lucas Matonya

**A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of
Master's in Mathematical and Computer Science and Engineering of the Nelson
Mandela African Institution of Science and Technology**

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ABSTRACT

Drug abuse remains to be the global burden causing a large number of death and disability, it is now termed as a significant threat to public health for both developed and developing countries. This work presents a mathematical model as a new approach towards understanding and controlling the drug abuse in Tanzania. The mathematical model to study the effect of drug abuse in the society is developed and analyzed in this study. The model's transmission process is considered as a social contact process between susceptible individuals and drug users based on the epidemiology principles. The model used the factors that were identified through feasibility study as the significant factors that control the dynamics of drug abused population in the particular society. The model is analyzed using next generation matrix method the number of new drug users that one drug user can produce in the entire period of abuse is computed and identifies as an epidemic threshold value, R_0 . Using R_0 the condition for existence and stability of stationary point is established. Using numerical simulation, the dynamical behavior of the model is explored and the result shows the significant contribution by the rate of adequate contact between the susceptible individual and the drug user, and the rate of recovery of drug user after rehabilitation as the major factors or players that dictate the dynamics of drug user population in the society. The model is finally analyzed to study the dynamical behavior of the drug abuse when control is applied, the result shows the drug users are minimized significantly. The result signifies the importance of early control of the drug abuse problem through establishment of the strict laws that will narrow the possibility of this bad practice in societies. It is then vividly justified that, key result arising from this model is that prevention is indeed better than cure.

DECLARATION

I, Fikiri Lucas Matonya , do hereby declare to the Senate of Nelson Mandela African Institute of Science and Technology that this dissertation contained therein is my own, original work that I am the sole author that it has neither been submitted or presented for similar degree award in any other institution.

Fikiri Lucas Matonya
Candidate's Name



Signature

08/07/2022

Date

The above declaration is confirmed by:

Prof. Dmitry Kuznetsov
Supervisor



Signature

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CERTIFICATION

The undersigned certify that they have read and found the dissertation acceptance by the Nelson Mandela African Institution of Science and Technology titled: “*Mathematical Modeling on the Effects of Drug Abuse to the Societies,*” in partial fulfillment of the requirements for the degree of Master’s in Mathematical and Computer Science and Engineering of the Nelson Mandela African Institution of Science and Technology.

Prof. Dmitry Kuznetsov
Supervisor



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08/07/2022
Date

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DEDICATION

This work is dedicated to the whole Matonya's family.

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LIST OF ABBREVIATIONS AND SYMBOLS

DFE	Disease Free Equilibrium.
EE	Endemic Equilibrium.
R_0	Basic Reproduction number.
NFSI	Normalized Forward Sensitivity Index.
COPD	Chronic Obstructive Pulmonary Disease.
UNODC	United Nations office on drugs and crimes.
HAART	Highly Active Anti-retroviral Therapy.
DPE	Drug Persistent Equilibrium.

CHAPTER ONE

INTRODUCTION

1.1 Background of the Problem

The reports on drug abuse depict that the large number of people in the world which is estimated to 300 millions are engaged in drugs in different ways. In most societies drugs affect youth who are the man power in almost all societies around the world. Studies and reports shows that among there drug dealers, over five percent of them are aged between fifteen to sixty four years (Degenhardt *et al.*, 2017).

1.1.1 Drug Abuse Concept

Drug(s) stand for any chemical substance which when used without the directive of the physician affect living organism. These substance(s) have got variety of use but mostly may be used for treatment of various health problem of a human being (Matowo, 2013). On the other hand, drugs are substances that produce various reactions or modify psychological and physiological functioning when taken into the human body (Benedetti, 2013).

The term drug abuse may be defined as the unlawful, illegitimate, prohibited, illegal, inappropriate, extreme or excessive use of drugs. Teens are increasing engaging in prescribed to relieve several pain and stimulant medication which treat condition like attention deficit disorder and narcolepsy World health organization (2011).

The most common drugs that are used for drug abused includes alcohol, amphetamines, nicotine, alcohol, barbiturates, Cannabis, cocaine, methaqualone, opium alkaloids, synthetic opioids and other of the kind. The drug abuse if are not controlled at the earliest stage may lead to significant effect to the society and to well being of the user (Matowo, 2013). The excessive use of drug may result to addiction which may also be termed as dependence or addiction of a particular drug. Drug addiction affect the psychological, social and physical well being of the human being and the injurious effect to the whole community (Organization *et al.*, 2004).

1.1.2 Classification of Drug Abuse

Substances abuse include both those which are not prohibited by law (legal) and those which are prohibited by law (criminal). The classification of drugs depends on individual's level of drug use and the interest of the classifying organization. Different bodies and institution differentiate drug abuse depending on the context of a particular country or environment. For example the study by Parry (2005) shows that in South Africa substances abused are classified into the heavily abused drugs, moderately abused drugs and those that are less frequently abused drugs. The many societies the most abused drugs include but not limited to all forms of alcohol, cigarettes and mandrax (methaqualone) which may be used in its own or combined by other drugs and many other. The moderately abused drugs include but not limited to cocaine, crack-cocaine and heroin and many other depending on the location and context. The least abused substances include but not limmited to opium, Rohypnol, ketamine and wellconal.

Noted that, although the classification drug a subjective base on the classifying institution. The drugs and the resulting effects still exist and they continue to affect millions of youth around the world. Studies show that regardless of the context and different amplitude this problem occur but all of these classification of drug use lead to different health, psychological and physical effect to the individuals and to the whole society at large (Borum *et al.*, 2003).

1.1.3 Drug Use as a Disease

Drug addiction is a disease that affect the brain that has a recognized cognitive behavioural and physiological characteristics which cause compulsive and continued use of drugs even with harmful consequences. According to Volkow and Li (2004), repeated use of drugs affect the normal functioning of a brain by affecting the natural inhibition and reward centers of the brain. Further more scientist have also found that the continued drug abuse changes for months or years the brain's anatomy and chemistry even after the individual has stopped using drugs. Volkow and Li (2004) published an example of the brain injury due to drug addiction that lasted for a very long time. Since the recovery of a drug addict takes a very long time, there is a high possibility for the drug addicts to relapse. Therefore the establishment control methods should take into consideration the period enough to produce and maintain behavioral change.

Drug addiction cause the affected individual not to be in control of his behavior and action, in most cases they fall in diverse behavior and action which may lead to health or legal consequences. In many society drug abuse in inseparable with the unlawful actions and behavior like robbery etc. Which if not controlled timely by the responsible institutions the society may end-up taking action to control the problem in their way of which in most cases these actions results to death of the drug users (Sproule, 2009).

Drug users, use different ways to administer drugs depending on the nature of the drugs one is using. The most popular ways of drug administration includes snorting, smoking and oral ingestion. The mode in which drug is being administered determine level of drug uptake in the blood and the related consequences (Rhodes *et al.*, 2011).

1.1.4 Drug Abuse Around the World

Different studies and report show the increasing trend of the abuse around th world. Whenever the drug abuse report is released whether in developed or developing countries one can be sure that the there will be an increase of drug users and volume of different drugs consumed (Sarangi *et al.*, 2008). This isn't a good sign as most of these drug users are youth or major community worker, thus the continual increase of drug user will not only affect their psychological, behavioral and actions but also the economic growth of the respective communities. Some very country like Afghanistan have managed to drop the drug use trend but since the drug abuse is more of the region or international problem therefore the local solution won't give the sustainable impact on the control of drug abuse in societies as it is vividly clear that drug abuse is stabilizing in developed countries which indirectly also affect the developing countries (Justice, 2015).

1.1.5 Drug Abuse Around Africa

Africa can not separate itself from the whole world, this means since other continents are affected by the drug abuse problem Africa is also affected significantly. Different studies justifies the use of some drugs like alcohol with poverty. Many countries in Africa like South Africa and Nigeria have portrayed a worse senatorial as the level of drug use in these countries is almost the peak of drug abuse in Africa and many other countries around the world (Mayosi *et al.*, 2012).

Although many no-government and governmental institution are available to fight for the drug abuse problem. But the result is not significant due to the fact that most of the drug dealers are powerful economically and some are even government officials. Drug dealing business is a high profit business which is the reason why many proposed solution do not yield the significant result as the people who are suppose to implement, normally end up being terminated or join the business. Africa is ranked number two in trafficking and using the drugs worldwide, the problem has resulted to a number of negative economic impact to the societies but also the enormous number of deaths due to drug abuse (Degenhardt *et al.*, 2017).

1.1.6 Drug Abuse in Tanzania

Tanzania is also among the African countries that are affected by drug abuse. The problem has lead to the number of effect to the society in particular youth who are the main workers in different Tanzania's society. Tanzania recorded a high peak of drug trafficking in the years between 1990 and 1995. The government has taken different measure like severe punishment to the drug dealers in order to eradicate the problem but still the number of drug users is increasing in societies.

The nation drug report shows the seasonal trend of drug use from 2001 to 2005. In 2005 Tanzania recorded a very significant drop of the use of there was decrease of use of cocaine of about 1.5 kilogram per year compared to the 7.5 kilogram per year recorded in 2001 (Hanson *et al.*, 2011)

It is certain that the drug abuse have significant negative effect to the society, it shows the undoubted effect to the people from the family level, national and the global level. In recent years the drug use has become a fashion among youth and if no effective way to solve drug abuse problem it will be a course of loss of millions of population in Tanzania and around the world.

1.2 Statement of the Problem

Literature and reports narrates that; Tanzania, like many other countries drug abuse mostly affect youth aged 15 years or above (Hanson *et al.*, 2011). One of the main reason that make Tanzania vulnerable to drug abuse is the presence of port, Tanzania is the center of transit goods from abroad to other countries like Rwanda, Malawi and Burundi just to name the few, this transit activities in a way influence an easy access and availability of drugs which are on transit to other countries surrounding Tanzania (Kilonzo-Nthenge *et al.*, 2013).

It is evident that presence of drug use have attracted many researcher to investigate and search and/or suggest solution for the problem. Regardless of the extensive work about the same but still the problem still persist and the major factors influencing its existence are still unknown. This justifies the need of the study that will thoroughly study the dynamics of drug abuse in the society and identify the factors that dictate its dynamics for a proper control plan.

This study intend to formulate a mathematical model system and analyze it to study the dynamics of drug abuse in the society and suggest the control mechanism. The model is analyzed to establish conditions for existence and extension of the drug abuse problem in the society, the local and global stability of stationary points and the system's dynamical behavior through numerical simulation.

1.3 Rationale of the Study

Drug abuse have existed for years, and its effect to the society is significant especially in urban ares or cities around the world. Different studies have been done to study the effect of drug use to the society. Very few of them uses mathematical approach to study the same but still we haven't reached the point where we can control the problem or explicitly explain its dynamics in the society. Therefore, the society lack formal information on drugs abuse and the effective ways to control the problem. Statistics shows the increase of drug induced effects at different levels depending on the kind of drugs taken by a particular individual. Drugs affect youth in almost every region in Tanzania for example in 2010, the large number of patients admitted at Mount Meru Hospital for drug abuse were aged between 15-24 years old (Kasilima, 2010). This clearly shows that there is a serious problem of drugs and requires a study that will thoroughly study the dynamics of drug abuse in the society, identify the factors affecting its dynamic and sagement the effective ways to control.

1.4 Objectives of the Study

This study is guided by the following objectives:

1.4.1 Main Objective

To analyze a mathematical model to study dynamics of drug abuse and effect in the society and suggest possible control methods.

1.4.2 Specific Objectives

- (i) To identify factors that affect drug abuse dynamics in the society.
- (ii) To formulate the basic model system for the dynamics of drug abuse to the society in Tanzania.
- (iii) To analyze the mathematical models for the effect of the drug abuse to the society in Tanzania.
- (iv) To explore the dynamical behavior, control(s) and effect of drug abuse in the society using numerical simulation.

1.5 Research Questions

In order to realize the specific objectives above we need to answer the following research questions:

- (i) Can the factors dictating the dynamics of drug abuse be identified?
- (ii) Can the mathematical models be developed using the identified variables and parameters?
- (iii) Under what conditions does the model feasible region and equilibrium states exist and stable?
- (iv) Does the theoretical result agree with the numerical simulations?

1.6 Significance of the Study

- (i) The study will add a body of knowledge about the drugs especially on the transmission and create awareness of drug abuse as a risk factor for diseases in our societies.
- (ii) The study will inform policy makers to decide on the best control mechanism to eradicate the problem or rather narrow the possibility of its occurrence, and also will help policy makers to regulate policies that can reduce that uses of drugs among people.
- (iii) The knowledge about the drug abuse will save people's life and to examine the preventive methods to avoid the effects of drugs.
- (iv) The study will alert the possible dangers of drugs consumption to our health and contribute knowledge to the academic world.
- (v) Suggest alternative control mechanism that if implemented effectively will also be a door way in fighting other infectious disease like HIV/AIDS transmission.

1.7 Delineation of the Study

In this drug use model, consider drug user population dynamics in the society, divide the human beings into six subgroups: first are the human being who have not yet started using drugs but may start to use drugs if they interact with I_1 or I_2 known as susceptible and symbolized by S , second are the people whose company and environment is risky, these are the people who have started using the drugs but they haven't declared or shown symptoms to be termed as light or heavily addicted users to be referred as exposed and denoted by E . Third and fourth are the people who are light and heavily addicted users who have shown symptoms and capable of influencing another susceptible individual to start using drugs to be referred as Light drug users and heavy drug users and denoted by I_1 and I_2 respectively, fifth are the the drug users who have started treatment in the rehabilitation centers to be referred to patient and denoted by Q and the last subgroup are the individuals if treated or through self reflection may recover and move to compartment R where by natural practice they are assumed to be temporary invulnerability from drug use before they become susceptible again. The model is formulated basing on the assumptions below:

- (i) Human being in subgroup S are enrolled at a constant rate.
- (ii) Same natural death rate for all individuals involved in this study.
- (iii) All individual involved in this study have equal chance of becoming drug user.
- (iv) On recovery individual become temporary invulnerable to drug abuse.
- (v) All individuals are born susceptible.
- (vi) Only heavy drug user may die due to drug addiction.
- (vii) Susceptible individual become drug user through social contact process only.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

This chapter presents a thoroughly review of various literature whose content are on drug abuse and its effect in the societies. The great number of studies investigates the effect of drug use in the societies. The literature was covered from various theme, research approach and methodologies, to investigate the work that has been done so far in understanding and counteracting the drug abuse problem in societies. Since mathematical modelling approach is used in this study we investigate extensively the studies that use mathematical modeling approach to study and understand the drug use problem in the context of Tanzania society and the world at large. Most of the studies link the drug abuse with other diseases like HIV, TB and other, this in a way affect the results that should be used to understand effect of drug abuse in the society and the level of exhaustiveness is low due to the broad coverage scope or multiple themes involved in the particular study. The study by Jado *et al.* (2012) modeled the effects of alcohol on HIV-infected patients the study focus on role of drug abuse to HIV patients under therapy. Pandrea *et al.* (2012) also depicted the role of alcohol in the transmission and progression of HIV. Mostly studies concetrate on the link that drug abuse has in the transmission and spread of other infectious and non-infectious diseases nut not the effect that drug abuse as a standalone problem has in the society.

Pandrea *et al.* (2010) also laid down the impact that drug abuse has in the transmission and progression of HIV. The study investigated different ways in which drug abuse affect the dynamic of HIV in the society. The result shows clearly that drug use may in a way foster the transmission of HIV virus, the rate of transmission increases as the level of drug abuse of a particular individual increases. Alcohol may also lead to an increase of virus production which consequently decreases the individual's immunity and as a result pave way to more secondary effect HIV virus has to a particular individual.

In Kenya, Pearson *et al.* (2014) unleashed the relationship between individuals with poor health due to alcohol and the the transmission and spread of HIV. The study investigated the role that HIV has in deterring the control and/or intervention taken to help the the individual who is largely affected by alcohol together with the relation it has with the number of HIV infection

and its induced mortality rate. The result also justifies the very significant interrelationship between the poor health of the alcohol addict and the transmission and spread of HIV .

The study by Takaidza *et al.* (2014) investigated on the optimal control mechanism of HIV virus in the community that has drug abuse. Several mathematical computation and the establishment of various parameters like basic reproduction number were performed to define the dynamics of HIV in the drug abuse infected communities. Pontryagin's maximum principle were used to establish the optimal control solution and the result narrated that when there is drug abuse in the community the cost of control plan and implementation gets higher compared to the drug free communities. This higher cost is due to the fact that people using drugs tend to expose them selves to the risky behavior and attitudes that may lead to transmission of HIV.

Tanzania like many other countries have laws and rules that ought to eradicate the drug practices in communities. The implementation of these roles and laws in most cases is not effective due to corruption and involvement into drug abuse business of people who were supposed implement there laws. Therefore regardless of the presence of these strict laws the drug dealers still exist with a significant increasing rate. In other side of the coin this send the message that the control approach currently being implemented is not effective as there isn't yet a clear study that has managed to lay down the actual causes of drug abuse in societies. Therefore using the literature and/or studies presented here it can be noted that the actual cause of the drug abuse in the society is not clear which results to noneffective ways to control the problem.

2.2 Observations From Reviewed Literature

The majority of reviewed literature investigated mostly about the effect that drug abuse have to the transmission of other diseases like HIV and TB. The results from these kind of studies are not exhaustive enough to answer question about the factor that influence drug abuse, population dynamics of drug users in the society and the effective control solutions to solve the drug abuse problem in the society. Other studies use approaches that do not give exhaustive results to understand the problem and control solutions. This study ought to study exhaustively the behavioral dynamics of drug abuse, major factor influencing the drug abuse problem and the effective control solution to solve the problem, using differential equations which is then analyzed to answer the targeted questions to a proper understanding and solution of the problem.

CHAPTER THREE

MATERIALS AND METHODS

3.1 Introduction

In this chapter present how the research was conducted systematically. It presents details of the research design, study site, data collection methods and data analysis methods. It also clarifies how the mathematical model was formulated and evaluated.

3.2 Research Design

The study adopted experimental research design. In modeling, the numerical experiments encompasses the collection of quantitative data that are put into quantitative analysis in a formal and stable way. It also includes inferential and simulation techniques to the study (Kothari, 2004). This approach best fitted to this study because the study used quantitative data to arrive at the solution of the problem.

3.3 Study Site

The study was conducted in four cities of Tanzania which are Dar es salaam, Arusha, Mbeya and Mwanza where purposely was assumed to have a significant number of drug users. The choice of these areas is also because the report on drug use in Tanzania shows that cities are largely affected than rural areas (Massele *et al.*, 1993).

3.4 Target Population

The study involved four cities in Tanzania which are Dar es salaam, Ausha, Mbeya and Mwanza. The target population are all human population aged fifteen years or more (≥ 15) in the four chosen cities of Tanzania who are at risk of using drug if they are exposed to the vulnerable environment.

3.5 Data Sources and Data Collection Methods

Secondary data sources were used for data collection for modeling. The data were gathered from publication, reports of data which was gathered by other authorities and/or institutions (Kothari, 2008).

3.6 Data Analysis Methods

In this study, the secondary data was used to estimate the model parameters. The parameters were then used in the mathematical analysis such as stability analysis and numerical simulation. MATLAB software tool was used in numerical simulations.

3.7 Materials and Tools

In this study, a mathematical model to study the dynamics of drug users and its effect to the society was formulated and analyzed. The model used differential equations to study the dynamical behaviour of the drug users in societies and hence help to provide the solution to control them. The basic properties of the model which are invariant region and positivity of solutions are established. Condition for stability and existence of equilibrium states are analyzed and established. The dynamical behavior by considering different aspects of the model were also analyzed using numerical simulation.

3.8 Model Development

3.8.1 Model Description

In this drug use model, consider drug user population dynamics in the society, we divide the human beings into six subgroups: first are the human being who have not yet started using drugs but may start to use drugs if they interact with I_1 or I_2 known as susceptible and symbolized by S , second are the people whose company and environment is risky, these are the people who have started using the drugs but they haven't declared or shown symptoms to be termed as light or heavily addicted users to be referred as exposed and denoted by E . Third and fourth are the people who are light and heavily addicted users who have shown symptoms and capable of influencing another susceptible individual to start using drugs to be referred as Light drug users and heavy drug users and denoted by I_1 and I_2 respectively, fifth are the the drug users who have started treatment in the rehabilitation centers to be referred to patient and denoted by Q and the last subgroup are the individuals if treated or through self reflection may recover and move to compartment R where by natural practice they are assumed to be temporary invulnerability from drug use before they become susceptible again.

3.8.2 Description of Interaction

The close relationship and/or frequency contact of susceptible human and the drug users I_1 and I_2 , increase the chances of becoming a drug user and thus become at high risky subgroup E at a rate ψ_1 and ψ_2 respectively. Extensive exposure to vulnerable drug use environment lead to light drug user I_1 or heavy drug user I_2 at a rate α_1 and proportional ρ or $(1 - \rho)$ respectively. With time the light drug user may become addict to drugs become heavy drug user at a rate $\rho_1\sigma_1$. With self reflection or by changing environment and company light drug addict may recover and become temporally invulnerable to drug use at a rate $\rho_3\sigma_1$, otherwise depending on the nature of the family and the environment they will be taken to the rehabilitation centers for treatment at the rate $\rho_2\sigma_1$. The heavy drug addicts may be taken to the rehabilitation centers for treatment at a rate σ_2 otherwise if not treated they die due to extensive drug addiction at a rate μ_2 . A successful treatment of drug user in the rehabilitation centers lead to recovery at a rate α_2 , the recovered group become temporally invulnerable to the drug use environment but with time they become susceptible again at a rate α_3 . Human beings are recruited at a constant rate λ and die naturally at the rate μ_1 . ω is the coefficient that represent a control effort using the method of early identification and quarantine of the drug for treatment to reduce the force of infection that exposes people in the community to drug abuse. Like any human behavior, drug usage varies across spectrum. So, for this modelling exercise, the definition of Light user and Heavy user was based simply upon the frequency of use. Heavy users are the people who used drugs at least weekly and Light users are all people who had used drugs in the last year. Clinicians and Researchers commonly divide drug consumption into three levels experimental use occasional and social consumption (Everingham, 1994).

3.8.3 Parameters Description

Table 1: Parameters Description for Drug Use Model

Parameter	Description
λ	The rate of recruitment of human population
ψ_1 & ψ_2	The rate that S become E due to I_1 and I_2 respectively
α_1	Progression rate out of E to drug user state.
ρ	Proportional of E becoming I_2
σ_1	Rate at which individual leave I_1
ρ_1	Proportional of I_1 becoming I_2
ρ_2	Proportional of I_1 becoming Q
ρ_3	Proportional of I_1 becoming R
α_2	The rate at which Q become R
α_3	The rate at which R become S
μ_1	Natural death rate of human being.
μ_2	Drug addiction induced death rate of human being.
ω	Control efforts to protect S
N	Total human population.

3.8.4 Model Assumptions

The model is formulated basing on the assumptions below:

- (i) Human being in subgroup S are enrolled at a constant rate.
- (ii) Same natural death rate for all individuals involved in this study.
- (iii) All individual involved in this study have equal chance of becoming drug user.
- (iv) On recovery individual become temporary invulnerable to drug abuse.
- (v) All individuals are born susceptible.
- (vi) Only heavy drug user may die due to drug addiction.
- (vii) Susceptible individual become drug user through social contact process only.

Using the description of interaction and assumption, we summarize the dynamics of drug use in the communities in a compartmental diagram presented in Fig. 1. The figure captures the interaction between the susceptible, drug users and individuals under treatment.

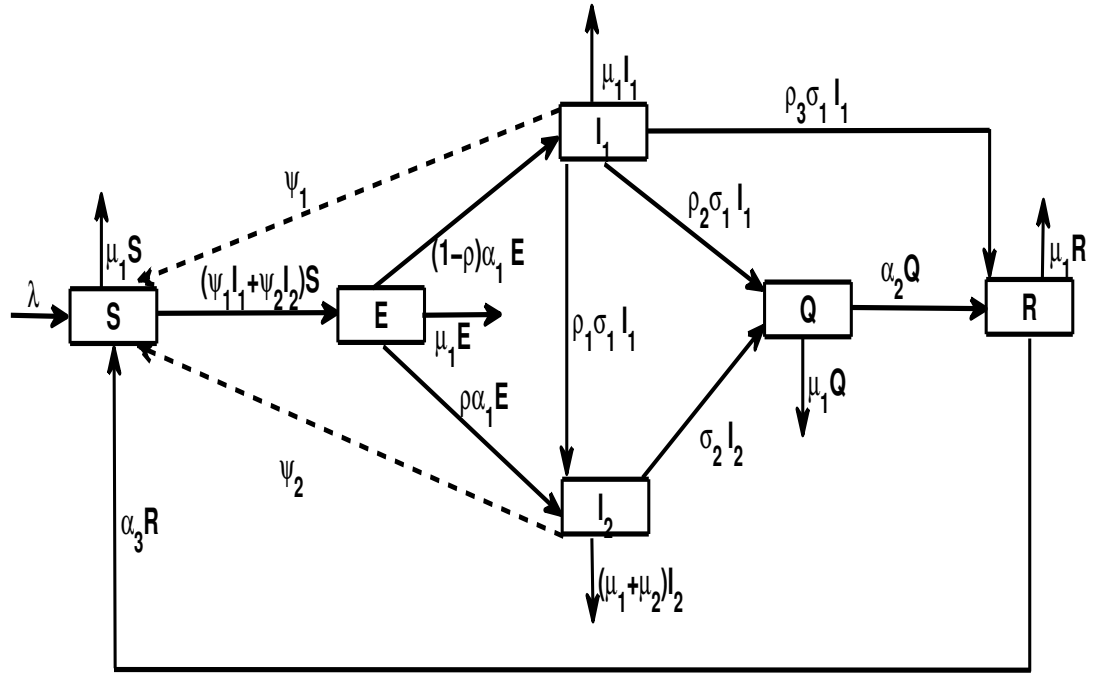


Figure 1: Compartmental Model for Drug Use

3.8.5 Model Equation

The dynamics of drug use in the society is presented by system (1)

$$\begin{aligned}
\frac{dS}{dt} &= \lambda + \alpha_3 R - (\psi_1 I_1 + \psi_2 I_2)(1 - \omega)S - \mu_1 S \\
\frac{dE}{dt} &= (\psi_1 I_1 + \psi_2 I_2)(1 - \omega)S - ((1 - \rho) + \rho)\alpha_1 E - \mu_1 E \\
\frac{dI_1}{dt} &= (1 - \rho)\alpha_1 E - (\rho_1 \sigma_1 + \rho_2 \sigma_1 + \rho_3 \sigma_1 + \mu_1)I_1 \\
\frac{dI_2}{dt} &= \rho \alpha_1 E + \rho_1 \sigma_1 I_1 - (\sigma_2 + \mu_1 + \mu_2)I_2 \\
\frac{dQ}{dt} &= \rho_2 \sigma_1 I_1 + \sigma_2 I_2 - (\alpha_2 + \mu_1)Q \\
\frac{dR}{dt} &= \rho_3 \sigma_1 I_1 + \alpha_2 Q - (\mu_1 + \alpha_3)R \\
\rho_1 + \rho_2 + \rho_3 &= 1
\end{aligned} \tag{1}$$

3.9 Analysis of the Model

Here analyze the basic drug abuse model (without control $\omega = 0$) is presented. Here the the mathematical meaningfulness, existence of stationary point and their stability are established. The basic reproduction number R_0 and effective reproduction number R_e are computed using next generation matrix method. Use R_0 to define the condition for global and local stability of the stationary points. In the numerical simulation section depict the dynamical behavior of the drug abuse model with and without control that aim at reducing the effect of drug abuse in the society.

Therefore in this section we consider the drug abuse model when $\omega = 0$ as in the system (2) below:

$$\begin{aligned}
\frac{dS}{dt} &= \lambda + \alpha_3 R - (\psi_1 I_1 + \psi_2 I_2 + \mu_1) S \\
\frac{dE}{dt} &= (\psi_1 I_1 + \psi_2 I_2) S - ((1 - \rho) + \rho) \alpha_1 E - \mu_1 E \\
\frac{dI_1}{dt} &= (1 - \rho) \alpha_1 E - (\rho_1 \sigma_1 + \rho_2 \sigma_1 + \rho_3 \sigma_1 + \mu_1) I_1 \\
\frac{dI_2}{dt} &= \rho \alpha_1 E + \rho_1 \sigma_1 I_1 - (\sigma_2 + \mu_1 + \mu_2) I_2 \\
\frac{dQ}{dt} &= \rho_2 \sigma_1 I_1 + \sigma_2 I_2 - (\alpha_2 + \mu_1) Q \\
\frac{dR}{dt} &= \rho_3 \sigma_1 I_1 + \alpha_2 Q - (\mu_1 + \alpha_3) R
\end{aligned} \tag{2}$$

3.10 Basic Properties of the Model

3.10.1 Invariant Region

Due to the fact that drugs affect human population, then, in order to model this process you need to assume that all of the defined state variables and parameters used in the model are non-negative for $\forall t \geq 0$. You are required to analyze the drug abuse model so that it is defined in a suitable feasible region which has positive state variables and therefore result to Theorem 3.1;

Theorem 3.1

The existing forward solutions in R_+^6 of the drug abuse model system are feasible $\forall t \geq 0$ whenever they enter the invariant region namely Λ .

where

$$\Lambda = (S, E, I_1, I_2, Q, R) \in R_+^6 : S + E + I_1 + I_2 + Q + R \leq N$$

and Λ is what defines as the positive invariant region of drug abuse model system

Proof. We are required to show that that all drug abuse model solution are feasible $\forall t > 0$ entering invariant region Λ .

We now let $\Lambda = (S, E, I_1, I_2, Q, R) \in R^6$ be solution space of the system with non-negative initial conditions.

The total human population is as given in (3)

$$N = S + E + I_1 + I_2 + Q + R. \tag{3}$$

Then:

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dE}{dt} + \frac{dI_1}{dt} + \frac{dI_2}{dt} + \frac{dQ}{dt} + \frac{dR}{dt} \quad (4)$$

Substituting system (2) into equation (4) we get,

$$\frac{dN}{dt} = \lambda - \mu_1 N - \mu_2 I_2 \quad (5)$$

Which then gives:

$$\frac{dN}{dt} \leq \lambda - \mu_1 N$$

and then;

$$\frac{dN}{dt} + \mu_1 N \leq \lambda$$

To solve this we need to find $IF = e^{\mu_1 t}$ then we multiply it through out to get the solution below;

$$e^{\mu_1 t} \frac{dN}{dt} + e^{\mu_1 t} N \mu_1 \leq (\lambda) e^{\mu_1 t}$$

this will then yield

$$\frac{d(Ne^{\mu_1 t})}{dt} \leq (\lambda) e^{\mu_1 t}$$

After integrating we get

$$Ne^{\mu_1 t} \leq \frac{(\lambda)}{\mu_1} e^{\mu_1 t} + C$$

this follows that;

$$N \leq \frac{(\lambda)}{\mu_1} + Ce^{-\mu_1 t}$$

Now if we consider the initial condition $t = 0, N(t = 0) = N_0$ and use it we obtain;

$$N_0 - \frac{(\lambda)}{\mu_1} \leq C$$

Which then gives;

$$N \leq \frac{(\lambda)}{\mu_1} + (N_0 - \frac{(\lambda)}{\mu_1}) e^{-\mu_1 t}$$

Now this means that when $N_0 > \frac{(\lambda)}{\mu_1}$, the population will be reduced asymptotically to $\frac{(\lambda)}{\mu_1}$ and when $N_0 < \frac{(\lambda)}{\mu_1}$ the human population increases asymptotically to $\frac{(\lambda)}{\mu_1}$.

This signify that all the feasible solutions of the drug abuse model system enter the region

$$\Lambda = \left\{ (S, E, I_1, I_2, Q, R) : N \leq \text{Max} \left\{ N_0, \frac{(\lambda)}{\mu_1} \right\} \right\}$$

□

3.10.2 Positivity of the Solution

Here is need to show that variables and parameters used in the drug abuse model are non negative $\forall t \geq 0$.

Theorem 3.2

Assume the initial values of the drug abuse model system (2) to be: $(S(0) > 0$ and $(E(0), I_1(0), I_2(0), Q(0), R(0)) \geq 0$. Then the model's solution set $S(t), E(t), Q(t), I_1(t), I_2(t)$ and $R(t)$ are positive $\forall t \geq 0$.

Proof. To do this, needs to consider each equation in the drug abuse model system 2.

Consider the first equation in the drug abuse model system,

$$\frac{dS}{dt} = \lambda + \alpha_3 R - (\psi_1 I_1 + \psi_2 I_2 + \mu_1) S \quad (6)$$

$$\geq -(\psi_1 I_1 + \psi_2 I_2 + \mu_1) S$$

$$\frac{dS}{dt} \geq -(\psi_1 I_1 + \psi_2 I_2 + \mu_1) S$$

Integration yields

$$S \geq S_0 e^{-\int_0^t (\psi_1 I_1 + \psi_2 I_2 + \mu_1) d\tau} > 0$$

since

$$e^{-\int_0^t (\psi_1 I_1 + \psi_2 I_2 + \mu_1) d\tau} > 0.$$

In the next equation we have:

$$\frac{dE}{dt} = (\psi_1 I_1 + \psi_2 I_2) S - ((1 - \rho) + \rho) \alpha_1 E - \mu_1 E. \quad (7)$$

Thus

$$\frac{dE}{dt} \geq -(\alpha_1 + \mu) E.$$

Integration yields

$$E \geq E_0 e^{-(\alpha_1 + \mu)t} > 0$$

since

$$e^{-(\alpha_1 + \mu)t} > 0.$$

Then consider

$$\frac{dQ}{dt} = \rho_2 \sigma_1 I_1 + \sigma_2 I_2 - (\alpha_2 + \mu_1) Q. \quad (8)$$

Thus

$$\frac{dQ}{dt} \geq -(\alpha_2 + \mu_1) Q.$$

After integration, yields

$$Q \geq Q e^{-(\alpha_2 + \mu_1)t} > 0.$$

since

$$e^{-(\alpha_2 + \mu_1)t} > 0.$$

Then for

$$\frac{dI_1}{dt} = (1 - \rho) \alpha_1 E - (\rho_1 \sigma_1 + \rho_2 \sigma_1 + \rho_3 \sigma_1 + \mu_1) I_1. \quad (9)$$

Will have

$$\frac{dI_1}{dt} \geq -(\sigma_1 + \mu_1) I_1.$$

Integrating you get

$$I_1 \geq I_{10} e^{-(\sigma_1 + \mu_1)t} > 0,$$

since

$$e^{-(\sigma_1 + \mu_1)t} > 0.$$

And for

$$\frac{dI_2}{dt} = \rho E + \rho_1 \sigma_1 I_1 - (\sigma_2 + \mu_1 + \mu_2) I_2. \quad (10)$$

Thus

$$\frac{dI_2}{dt} \geq -(\sigma_2 + \mu_1 + \mu_2) I_2.$$

After Integration, get

$$I_2 \geq I_{20} e^{-(\sigma_2 + \mu_1 + \mu_2)t} > 0,$$

since

$$e^{-(\sigma_2 + \mu_1 + \mu_2)t} > 0.$$

For the last equation for the system (2)

$$\frac{dR}{dt} = \rho_3 \sigma_1 I_1 + \alpha_2 Q - (\mu_1 + \alpha_3) R \quad (11)$$

Thus

$$\frac{dR}{dt} \geq -(\mu_1 + \alpha_3)R.$$

Integrating, get

$$R \geq R_0 e^{-(\mu_1 + \alpha_3)t} > 0,$$

since

$$e^{-(\mu_1 + \alpha_3)t} > 0.$$

□

3.11 Model Analysis

This part of the study present the existence of stationary states, the number of drug use cases produced by one drug user in the whole drug user period and stability of the stationary points.

3.11.1 Drug Use Free Equilibrium

Obtain the Drug Use Free equilibrium by setting $E = Q = I_1 = I_2$ and $R = 0$ and substitute to the drug abuse model system (2).

The drug use free-equilibrium point of the drug abuse model system is as given in (12)

$$X_0(S^0, E^0, I_1^0, I_2^0, Q^0, R^0) = \left(\frac{\lambda}{\mu_1}, 0, 0, 0, 0, 0 \right) \quad (12)$$

3.11.2 Basic Reproduction Number R_0 for Drug Abuse

This is the number of secondary drug abuse cases that are to be produced by one drug user in the whole drug abuse period of that particular individual in a population defined by only susceptible population. The criteria for this dimensionless parameter is that if $R_0 < 1$, then the single drug user in a population defined by only susceptible population may influence less than one individual to start using drugs. This entails that drug abuse problem may be eradicated from the population and the drug free stationary point is asymptotically stable which also means that the drug abuse cannot attack the society.

When $R_0 > 1$ it portray that one drug user in a population defined by only susceptible population may influence more than one individuals to start using drugs. This also means that drug abuse

problem may hence the drug abuse may continue to stay in the society. This situation also means that the drug free equilibrium point is unstable and that it is vividly clear that the drug abuse problem can attack the society and stay for a long time.

And if $R_0 = 1$ it portray that one drug user in entirely susceptible population influence drug use to one new human being. Hence the drug abuse problem will be alive in the society without an serious epidemic as narrated by Allen *et al.* (2008).

To find the basic reproduction number use next generation method by Van den Driessche and Watmough (2002). Consider a heterogeneous population in compartments S, E, I_1, I_2, Q and R arranged such that m drug using classes come first.

Assume $F_i(x)$ as rate of entrance of new drug users in class i , $V_i^+(x)$ rate of transfer of individuals in the class i by any other means except the drug abuse induced $V_i^-(x)$ be the rate of transfer of individuals out of class i .

The the model system is as presented below:

$$x_i' = F_i(x) - V_i(x) \quad (13)$$

where $V_i(x) = V_i^-(x) - V_i^+(x)$.

Then use x_0 , to find the $m \times m$ matrices F and V

$$F = \left(\frac{\partial F_i}{\partial x_j}(x_0) \right), \quad V = \left(\frac{\partial V_i}{\partial x_j}(x_0) \right) \quad (14)$$

with $1 \leq i, j \leq m$.

By using the study by Diekmann *et al.* (1990) call Matrix FV^{-1} , a next generation matrix Thus:

$$R_0 = \rho(FV^{-1}) \quad (15)$$

$$\begin{aligned}
\frac{dE}{dt} &= (\psi_1 I_1 + \psi_2 I_2)S - ((1 - \rho) + \rho)\alpha_1 E - \mu_1 E \\
\frac{dI_1}{dt} &= (1 - \rho)\alpha_1 E - (\rho_1 \sigma_1 + \rho_1 \sigma_1 + \rho_1 \sigma_1 + \mu_1)I_1 \\
\frac{dI_2}{dt} &= \rho E + \rho_1 \sigma_1 I_1 - (\sigma_2 + \mu_1 + \mu_2)I_2 \\
\frac{dQ}{dt} &= \rho_2 \sigma_1 I_1 + \sigma_2 I_2 - (\alpha_2 + \mu_1)Q \\
\frac{dR}{dt} &= \rho_3 \sigma_1 I_1 + \alpha_2 Q - (\mu_1 + \alpha_3)R \\
\frac{dS}{dt} &= \lambda + \alpha_3 R - (\psi_1 I_1 + \psi_2 I_2 + \mu_1)S
\end{aligned} \tag{16}$$

Now from the system (16) the drug using classes are the one with compartment E, I_1, I_2 and Q , this will now yield

$$\mathbf{F}_i = \begin{pmatrix} (\psi_1 I_1 + \psi_2 I_2)S \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{17}$$

And

$$\mathbf{V}_i = \begin{pmatrix} \alpha_1 E + \mu_1 E \\ (\rho_1 \sigma_1 + \mu_1)I_1 - (1 - \rho)\alpha_1 E \\ (\sigma_2 + \mu_1 + \mu_2)I_2 - \rho E - \rho_1 \sigma_1 I_1 \\ (\alpha_2 + \mu_1)Q - \rho_2 \sigma_1 I_1 - \sigma_2 I_2 \end{pmatrix}. \tag{18}$$

The computation for matrices F and V at x_0 are as given below:

$$\frac{\partial F_i}{\partial x_j} = \begin{pmatrix} \frac{\partial F_1}{\partial E} & \frac{\partial F_1}{\partial I_1} & \frac{\partial F_1}{\partial I_2} & \frac{\partial F_1}{\partial Q} \\ \frac{\partial F_2}{\partial E} & \frac{\partial F_2}{\partial I_1} & \frac{\partial F_2}{\partial I_2} & \frac{\partial F_2}{\partial Q} \\ \frac{\partial F_3}{\partial E} & \frac{\partial F_3}{\partial I_1} & \frac{\partial F_3}{\partial I_2} & \frac{\partial F_3}{\partial Q} \\ \frac{\partial F_4}{\partial E} & \frac{\partial F_4}{\partial I_1} & \frac{\partial F_4}{\partial I_2} & \frac{\partial F_4}{\partial Q} \end{pmatrix} = \begin{pmatrix} 0 & \psi_1 S & \psi_2 S & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now at x_0 will have

$$\mathbf{F} = \begin{pmatrix} 0 & \psi_1 \frac{\lambda}{\mu_1} & \psi_2 \frac{\lambda}{\mu_1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{19}$$

and for V will have:

$$V = \frac{\partial V_i}{\partial x_j}(x_0)$$

$$\mathbf{V} = \begin{pmatrix} \alpha_1 + \mu_1 & 0 & 0 & 0 \\ -(1 - \rho)\alpha_1 & (\rho_1\sigma_1 + \mu_1) & 0 & 0 \\ -\rho\alpha_1 & -\rho_1\sigma_1 & (\sigma_2 + \mu_1 + \mu_2) & 0 \\ 0 & -\rho_2\sigma_1 & -\sigma_2 & (\alpha_2 + \mu_1) \end{pmatrix}. \quad (20)$$

Can obtain V^{-1} and FV^{-1} using maple, then the spectrol radius which is also the basic repro-
duction number is as in (21) :

$$R_0 = \frac{((\eta\psi_1 + \rho\psi_2)\mu_1 + \psi_2\rho_1(\eta + \rho)\sigma_1 + \eta\psi_1(\sigma_2 + \mu_2))\alpha_1\lambda}{\mu_1(\alpha_1 + \mu_1)(\sigma_2 + \mu_1 + \mu_2)(\rho_1\sigma_1 + \mu_1)} \quad (21)$$

Where $\eta = (1 - \rho)$.

When early identification and quarantine control method is applied, the number of secondary cases of drug users will decrease which as a result will reduce the number of adequate contact between the drug user and the susceptible population. Now using the same next generation matrix method can generate an effective reproduction number that consider the effect of control as in (22)

$$R_e = \frac{((\eta\psi_1 + \rho\psi_2)\mu_1 + \psi_2\rho_1(\eta + \rho)\sigma_1 + \eta\psi_1(\sigma_2 + \mu_2))(1 - \omega)\alpha_1\lambda}{\mu_1(\alpha_1 + \mu_1)(\sigma_2 + \mu_1 + \mu_2)(\rho_1\sigma_1 + \mu_1)} \quad (22)$$

Where $\eta = (1 - \rho)$.

When $\omega \neq 0$ it reduces the value of the number of secondary cases produced as it can be seen in the expression (22), it can be seen that the increase of the control efforts will reduce the value of the expression in (22) which the number of secondary cases of drug users produced by a drug user in the entire period of drug abuse, but when $\omega = 0$ the expression (22) become the basic reproduction number as given in (21).

3.12 Steady States and Local Stability of the Critical Points

This section presents the conditions for stability of critical points of the drug abuse model system (2).

3.12.1 Disease Free Equilibrium

The model has a drug free equilibrium in which we set all infectious compartment and the derivatives equal to zero. Then we have the drug free-equilibrium point is as given in (23)

$$X_0(S^0, E^0, I_1^0, I_2^0, Q^0, R^0) = \left(\frac{\lambda}{\mu_1}, 0, 0, 0, 0, 0 \right). \quad (23)$$

3.12.2 Local Stability of the Drug-Free Equilibrium Point

This section presents the analysis for local stability of the drug free stationary point of the drug abuse model. The use Jacobian method by considering that all equations in drug abuse model in (2) are analyzed at the drug free stationary point X_0 . Are required to compute and asses the eigenvalues of Jacobian matrix in order to verify that the drug free stationary point is locally and asymptotically stable. Further more is needs to show that the real parts of the eigenvalues of the matrix at X_0 are negative.

Using the concept by Martcheva (2015), are required to show that eigenvalues are negative, in which need to prove that determinant of the Jacobian matrix is positive and its trace negative. .

Now using the Jacobian matrix of the system (2) at X_0 can prove that, drug free stationary point E^0 is locally asymptotically stable and leads to the following theorem:

$$\mathbf{J}(X_0) = \begin{pmatrix} -\mu_1 & 0 & -\frac{\psi_1 \lambda}{\mu_1} & -\frac{\psi_2 \lambda}{\mu_1} & 0 & \alpha_3 \\ 0 & -(\alpha_1 + \mu_1) & \frac{\lambda \psi_1}{\mu_1} & \frac{\psi_2 \lambda}{\mu_1} & 0 & 0 \\ 0 & (1 - \rho)\alpha_1 & -(\sigma_1 + \mu_1) & 0 & 0 & 0 \\ 0 & \rho\alpha_1 & \rho_1\sigma_1 & -(\sigma_2 + \mu_1 + \mu_2) & 0 & 0 \\ 0 & 0 & \rho_2\sigma_1 & \sigma_2 & -(\alpha_2 + \mu_1) & 0 \\ 0 & 0 & \rho_3\sigma_1 & 0 & \alpha_2 & -(\mu_1 + \alpha_3) \end{pmatrix} \quad (24)$$

Using trace and determinant method: need to check if trace is negative and the determinant of matrix (24) is positive.

Computing, get trace of the matrix (24) as given below:

$$\mathbf{Trace} = -\mu - (\alpha_1 + \mu_1) - (\sigma_1 + \mu_1) - (\sigma_2 + \mu_1 + \mu_2) - (\alpha_2 + \mu_1) - (\omega + \mu) - (\mu_1 + \alpha_3) \quad (25)$$

Therefore the trace of a matrix (24) in negative.

Then compute determinant of the matrix (24) in which it is positive if

$$\frac{((\eta \psi_1 + \rho \psi_2) \mu_1 + \psi_2 \rho_1 (\eta + \rho) \sigma_1 + \eta \psi_1 (\sigma_2 + \mu_2)) \alpha_1 \lambda}{\mu_1 (\alpha_1 + \mu_1) (\sigma_2 + \mu_1 + \mu_2) (\rho_1 \sigma_1 + \mu_1)} < 1$$

where

$$\frac{((\eta \psi_1 + \rho \psi_2) \mu_1 + \psi_2 \rho_1 (\eta + \rho) \sigma_1 + \eta \psi_1 (\sigma_2 + \mu_2)) \alpha_1 \lambda}{\mu_1 (\alpha_1 + \mu_1) (\sigma_2 + \mu_1 + \mu_2) (\rho_1 \sigma_1 + \mu_1)} \quad (26)$$

represents the basic reproduction number, R_0 .

The above results justifies that the drug free stationary point x^0 is locally asymptotically stable as in theorem below:

Theorem 3.3

The Drug Free stationary point X_0 is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

3.12.3 Global Stability of the Drug-Free Equilibrium Point

This section presents the analysis of the global stability of the drug free stationary point. Use the method by Castillo-Chavez *et al.* (2002) known as Metzler matrix as presented in the steps below:

Assume \mathbf{Y}_n to be the vector of compartments which can not influence any drug user, \mathbf{Y}_i to be the vector for classes that can influence individuals to start using drugs and $\mathbf{Y}_{X_0,n}$ to be the vector of drug free stationary point.

$$\begin{cases} \frac{d\mathbf{Y}_n}{dt} = A_1(\mathbf{Y}_n - \mathbf{Y}_{X_0,n}) + A_2\mathbf{Y}_i \\ \frac{d\mathbf{Y}_i}{dt} = A_3\mathbf{Y}_i \end{cases} \quad (27)$$

this will then yield:

$$\mathbf{Y}_n = (S, R, E, Q)^T \quad \mathbf{Y}_i = (I_1, I_2) \quad \mathbf{Y}_{X_0,n} = \left(\frac{\phi}{\mu}, 0\right) \quad (28)$$

$$\mathbf{Y}_n - \mathbf{Y}_{X_0,n} = \begin{pmatrix} S - \frac{\lambda}{\mu_1} \\ R \\ E \\ Q \end{pmatrix} \quad (29)$$

Using the Metzler matrix method ,required to show that the eigenvalues of a matrix A_1 are negative and A_3 is a Metzler matrix which represent a matrix with non-negative off diagonal

element. The subsystem (27) then becomes:

$$\begin{pmatrix} \lambda + \alpha_3 R - (\psi_1 I_1 + \psi_2 I_2 + \mu_1) S, \\ \rho_3 \sigma_1 I_1 + \alpha_2 Q - (\mu_1 + \alpha_3) R. \\ (\psi_1 I_1 + \psi_2 I_2) S - ((1 - \rho) + \rho) \alpha_1 E - \mu_1 E. \\ \rho_2 \sigma_1 I_1 + \sigma_2 I_2 - (\alpha_2 + \mu_1) Q. \end{pmatrix} = A_1 \begin{pmatrix} S - \frac{\vartheta}{\mu} \\ R \\ E \\ Q \end{pmatrix} + A_2 \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \quad (30)$$

and

$$\begin{pmatrix} (1 - \rho) \alpha_1 E - (\rho_1 \sigma_1 + \rho_2 \sigma_1 + \rho_3 \sigma_1 + \mu_1) I_1, \\ \rho \alpha_1 E + \rho_1 \sigma_1 I_1 - (\sigma_2 + \mu_1 + \mu_2) I_2. \end{pmatrix} = A_3 \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \quad (31)$$

Again use the drug use influencing and non-influencing element from the general drug abuse model model to get matrices (32):

$$A_1 = \begin{pmatrix} -\mu_1 & \alpha_3 & 0 & 0 \\ 0 & -(\mu_1 + \alpha_3) & 0 & \alpha_2 \\ 0 & 0 & -(\alpha_1 + \mu_1) & 0 \\ 0 & 0 & 0 & -(\alpha_2 + \mu_1) \end{pmatrix} \quad (32)$$

$$A_2 = \begin{pmatrix} -\frac{\lambda \psi_1}{\mu_1} & -\frac{\lambda \psi_2}{\mu_1} \\ \rho_3 \sigma_1 & 0 \\ \frac{\lambda \psi_1}{\mu_1} & \frac{\lambda \psi_2}{\mu_1} \\ \rho_2 \sigma_1 & \sigma_2 \end{pmatrix} \quad (33)$$

$$A_3 = \begin{pmatrix} -(\sigma_1 + \mu_1) & 0 \\ \rho_1 \sigma_1 & -(\sigma_2 + \mu_1 + \mu_2) \end{pmatrix} \quad (34)$$

It is clearly seen that eigenvalues values of a matrix A_1 , are real and negative. This justifies that the system

$$\frac{d\mathbf{Y}_n}{dt} = A_1(\mathbf{Y}_n - \mathbf{Y}_{\mathbf{X}_0, n}) + A_2 \mathbf{Y}_i \quad (35)$$

is globally and asymptotically stable at $\mathbf{Y}_{\mathbf{X}_0}$.

Can also be seen that: A_3 has non-negative off-diagonal elements which means it is a Metzler stable matrix. Therefore Drug Free Equilibrium point for drug abuse model system is globally asymptotically stable as in the theorem below:

Theorem 3.4

The drug-free equilibrium point is globally asymptotically stable in E_0 if $R_0 < 1$ and unstable if $R_0 > 1$.

3.12.4 Existence of Drug Abuse Endemic Equilibrium

In this section establish the conditions for existence of the drug abuse endemic equilibrium point of the system (2).

To obtain the drug abuse endemic equilibrium point $E^*(S^*, E^*, I_1^*, I_2^*, Q^*, R^*)$ we set system (2) equal to zero and solve equations.

The model system (36) which exist for $R_0 > 1$.

$$\begin{aligned}
 \lambda + \alpha_3 R - (\psi_1 I_1 + \psi_2 I_2 + \mu_1) S &= 0 \\
 (\psi_1 I_1 + \psi_2 I_2) S - ((1 - \rho) + \rho) \alpha_1 E - \mu_1 E &= 0 \\
 (1 - \rho) \alpha_1 E - (\rho_1 \sigma_1 + \rho_2 \sigma_1 + \rho_3 \sigma_1 + \mu_1) I_1 &= 0 \\
 \rho \alpha_1 E + \rho_1 \sigma_1 I_1 - (\sigma_2 + \mu_1 + \mu_2) I_2 &= 0 \\
 \rho_2 \sigma_1 I_1 + \sigma_2 I_2 - (\alpha_2 + \mu_1) Q &= 0 \\
 \rho_3 \sigma_1 I_1 + \alpha_2 Q - (\mu_1 + \alpha_3) R &= 0
 \end{aligned} \tag{36}$$

Studies by Tumwiine *et al.* (2007) and that of Massawe *et al.* (2015) portray how to prove the existence the drug abuse endemic equilibrium points in which we are required to satisfy the condition $E \neq 0$ or $Q \neq 0$ or $I_1 \neq 0$ or $I_2 \neq 0$ that is $S > 0$ or $E > 0$ or $Q > 0$ or $I_1 > 0$ or $I_2 > 0$ or $R > 0$. Adding the equation in the system (36) yields (37)

$$\lambda - \mu_1(S + E + I_1 + I_2 + Q + R) - \mu_2 I_2 = 0 \tag{37}$$

We then have

$$\lambda = \mu_1 N + \mu_2 I_2 \tag{38}$$

Therefore since $\lambda > 0$, $\mu_1 > 0$ and $\mu_2 > 0$ we can discern that $\mu_1 N > 0$ and $\mu_2 I_2 > 0$ implying that $S > 0$, $E > 0$, $I_1 > 0$, $I_2 > 0$, $Q > 0$ and $R > 0$.

This prove that the drug abuse endemic equilibrium point of the drug abuse model system exists.

3.12.5 Local Stability of Drug Abuse Endemic Equilibrium Point

Theorem 3.5

The Drug Abuse Endemic equilibrium Point the system (2) is locally asymptotically stable whenever $R_0 > 1$

Proof. A disease is endemic in a population if it persists in the population. We investigate the stability of the endemic equilibrium using the trace and the determinant. The Jacobian matrix at $E^*(S^*, E^*, I_1^*, I_2^*, Q^*, R^*)$ is as given in (39).

$$\mathbf{J}(E^*) = \begin{pmatrix} -z_3 & 0 & -\psi_1(1-\omega)S^* & -\psi_2(1-\omega)S^* & 0 & \alpha_3 \\ z_4 & -z_5 & \psi_1(1-\omega)S^* & \psi_2(1-\omega)S^* & 0 & 0 \\ 0 & (1-\rho)\alpha_1 & -z_1 & 0 & 0 & 0 \\ 0 & \rho\alpha_1 & \rho_1\sigma_1 & -z_2 & 0 & 0 \\ 0 & 0 & \rho_2\sigma_1 & \sigma_2 & -(\alpha_2 + \mu_1) & 0 \\ 0 & 0 & \rho_3\sigma_1 & 0 & \alpha_2 & -(\mu_1 + \alpha_3) \end{pmatrix} \quad (39)$$

where:

$$\begin{aligned} z_1 &= ((\rho_1 + \rho_2 + \rho_3)\sigma_1 + \mu_1), \\ z_2 &= (\sigma_2 + \mu_1 + \mu_2), \\ z_3 &= ((\psi_1 I_1^* + \psi_2 I_2^*)(1-\omega) + \mu_1), \\ z_4 &= (\psi_1 I_1^* + \psi_2 I_2^*)(1-\omega), \\ z_5 &= ((1-\rho) + \rho)\alpha_1 + \mu_1 \\ z_6 &= \psi_1(1-\omega)S^* \\ z_7 &= \psi_2(1-\omega)S^* \\ z_8 &= (1-\rho)\alpha_1 \end{aligned}$$

From the Jacobian Matrix $J(E^*)$ after the computation and simplification yields the trace and determinant of the Jacobian matrix (39) as given in (40) and (41) respectively.

$$\text{Trace}(J(E^*)) = -(z_3 + z_5 + z_1 + z_2 + (\alpha_2 + \mu_1) + (\mu_1 + \alpha_3)) \quad (40)$$

$$\begin{aligned} \text{Det}(J(E^*)) &= (-z_3 - z_4)(\rho_1\sigma_1 z_7 + z_2 z_6) z_8 - (-\rho\alpha_1 z_7 z_4 + z_3(\rho\alpha_1 z_7 \\ &\quad - z_2 z_5)) z_1 \mu_1^2 + ((-z_3 - z_4)(\rho_1\sigma_1 z_7 + z_2 z_6)\alpha_2 + (\rho_1\alpha_4\sigma_1 z_7 \\ &\quad - z_2(\alpha_4\rho_3\sigma_1 - z_6\alpha_3)) z_4 - \alpha_3 z_3(\rho_1\sigma_1 z_7 + z_2 z_6)) z_8 \\ &\quad - (\alpha_2 + \alpha_3)(\rho\alpha_1 z_7 z_4 + z_3(\rho\alpha_1 z_7 - z_2 z_5)) z_1 \mu_1 \quad (41) \\ &\quad - \alpha_2(((-\rho_1\alpha_3\sigma_1 z_7 + (-z_6\alpha_3 + \alpha_3\sigma_1(\rho_2 + \rho_3)) z_2 + \rho_1\sigma_1\sigma_2) z_4 \\ &\quad + z_3(\rho_1\sigma_1 z_7 + z_2 z_6)) z_8 + z_1(\rho\alpha_1(\sigma_2 - z_7) z_4 \\ &\quad + z_3(\rho\alpha_1 z_7 - z_2 z_5))) \alpha_3 \end{aligned}$$

The determinant $Det(J(E^*))$ is positive if the condition (42) below is satisfied.

$$\begin{aligned}
& (z_8(\rho_1 \sigma_1 z_1 + z_2 z_6)z_4 + z_3 z_5 z_2 z_1) \mu_1^2 + (\alpha_2 + \alpha_3) ((\rho \alpha_1 z_1 \\
& + \rho_1 \sigma_1 z_2)z_7 + z_2 z_6 z_8)z_4 + z_3 z_5 z_2 z_1) \alpha_3 \alpha_2 \\
& > \\
& (((\mu_1 + \alpha_3)(\rho_1 \sigma_1 z_2 + z_2 z_6) \alpha_2 + ((\rho_1 \sigma_1 z_7 + z_2 z_6) \mu_1 + \alpha_3(\rho_1 \sigma_1 z_7 \\
& + 2z_2 z_6)) \mu_1)z_8 + \rho \alpha_1 z_1 z_7 (\mu_1 + \alpha_3)(\mu_1 + \alpha_2))z_3 + z_4 \alpha_3 (\sigma_1 ((\sigma_2 \rho_1 \\
& + z_2(\rho_2 + \rho_3)) \alpha_2 + \rho_3 z_2 \mu_1)z_8 + \rho \alpha - 1 \alpha_2 \sigma_2 z_1)
\end{aligned} \tag{42}$$

In the section above, Have shown that the drug use endemic equilibrium point $E^*(S^*, E^*, I_1^*, I_2^*, Q^*, R^*)$ of the drug abuse system (2) exist when $R_0 > 1$. Since trace and determinant of the Jacobian matrix (39) are function of $(S^*, E^*, I_1^*, I_2^*, Q^*, R^*)$ and $Trace(J(E^*)) < 0$ and $Det(J(E^*)) > 0$ given that the inequality condition (42) holds. Concludes that the drug abuse endemic equilibrium of system (2) is locally asymptotically stable whenever $R_0 > 1$. \square

The local stability of the drug use equilibrium implies that for a small perturbation of the drug use endemic equilibrium, the solutions of drug abuse system (2) will always converge to the point $E^*(S^*, E^*, I_1^*, I_2^*, Q^*, R^*)$ whenever $R_0 > 1$. Epidemiological, it implies that if a few individuals with drug use behavior are introduced in a fully susceptible population with $R_0 > 1$, then the drug use behavior would persist in the population.

3.12.6 Global Stability of Drug Abuse Endemic Equilibrium Point

This section analyzes the condition for stability of the drug abuse endemic equilibrium points. The results in the study by Van den Driessche and Watmough (2002), shows that the local stability of the Drug Free Equilibrium implies that the local stability of the Drug Abuse Endemic Equilibrium is also locally stable for the reverse condition.

Use Korobeinikov approach as described by Van den Driessche and Watmough (2002), Korobeinikov (2004) and Korobeinikov (2007) to find the global stability of Drug Abuse Endemic stationary point.

Thus formulate a suitable Lyapunov function for drug abuse model system in the following form:

$$V = \sum a_i (y_i - y_i^* \ln y_i) \tag{43}$$

where a_i is an appropriately selected positive constant, y_i is a population of the i^{th} class type, and y_i^* is the stationary point.

Will then have

$$V = W_1(S - S^* \ln S) + W_2(E - E^* \ln E) + W_4(I_2 - I_1^* \ln I_1) + W_5(I_2 - I_2^* \ln I_2) + W_3(Q - Q^* \ln Q) + W_6(R - R^* \ln R) \quad (44)$$

The constants W_i are non-negative in Λ such that $W_i > 0$ for $i = 1, 2, 3, \dots, 6$. V which is the Lyapunov function W_1, W_2, \dots, W_6 , defined as the Lyapunov function constant are selected so that V should be continuous and differentiable in a space.

Finding the time derivative of V get:

$$\frac{dV}{dt} = W_1(1 - \frac{S^*}{S}) \frac{dS}{dt} + W_2(1 - \frac{E^*}{E}) \frac{dE}{dt} + W_3(1 - \frac{I_1^*}{I_1}) \frac{dI_1}{dt} + W_4(1 - \frac{I_2^*}{I_2}) \frac{dI_2}{dt} + W_5(1 - \frac{Q^*}{Q}) \frac{dQ}{dt} + W_6(1 - \frac{R^*}{R}) \frac{dR}{dt} \quad (45)$$

Substituting the respective equations from the drug abuse model system (2) get:

$$\begin{aligned} \frac{dV}{dt} = & W_1(1 - \frac{S^*}{S})[\lambda + \alpha_3 R - (\psi_1 I_1 + \psi_2 I_2 + \mu_1)S] \\ & + W_2(1 - \frac{E^*}{E})[(\psi_1 I_1 + \psi_2 I_2)S - ((1 - \rho) + \rho)\alpha_1 E - \mu_1 E] \\ & + W_3(1 - \frac{I_1^*}{I_1})[(1 - \rho)\alpha_1 E - (\rho_1 \sigma_1 + \rho_2 \sigma_1 + \rho_3 \sigma_1 + \mu_1)I_1] \\ & + W_4(1 - \frac{I_2^*}{I_2})[\rho \alpha_1 E + \rho_1 \sigma_1 I_1 - (\sigma_2 + \mu_1 + \mu_2)I_2] \\ & + W_5(1 - \frac{Q^*}{Q})[\rho_2 \sigma_1 I_1 + \sigma_2 I_2 - (\alpha_2 + \mu_1)Q] \\ & + W_6(1 - \frac{R^*}{R})[\rho_3 \sigma_1 I_1 + \alpha_2 Q - (\mu_1 + \alpha_3)R] \end{aligned} \quad (46)$$

Computing time derivative of V at drug abuse endemic stationary point yields:

$$\begin{aligned} \frac{dV}{dt} = & -W_1(1 - \frac{S^*}{S})^2 - W_2(1 - \frac{E^*}{E})^2 - W_3(1 - \frac{I_1^*}{I_1})^2 \\ & - W_4(1 - \frac{I_2^*}{I_2})^2 - W_5(1 - \frac{Q^*}{Q})^2 - W_6(1 - \frac{R^*}{R})^2 \\ & + F(S, E, I_1, I_2, Q, R) \end{aligned} \quad (47)$$

where the function $F(S, E, I_1, I_2, Q, R)$ is non positive,

Using the insight from the study by McCluskey (2006) and Korobeinikov and Wake (2002).

You have

$F(S, E, I_1, I_2, Q, R) \leq 0$ for all S, E, I_1, I_2, Q, R , Then $\frac{dV}{dt} \leq 0$ for all S, E, I_1, I_2, Q, R and it is zero when $S = S^*, E = E^*, Q = Q^*, I_1 = I_1^*, I_2 = I_2^*, R = R^*$ Hence the largest compact invariant set in S, E, I_1, I_2, Q, R such that $\frac{dV}{dt} = 0$ is the singleton E^* which is Drug Abuse Endemic stationary point of the model system (2).

By LaSalle's invariant principle in La Salle (1976) we proclaim that E^* is globally asymptotically stable in the interior of drug abuse model system region of S, E, I_1, I_2, Q, R as in Theorem 3.6

Theorem 3.6

If $R_0 > 1$ then the drug abuse model system (2) has a distinctive drug abuse endemic equilibrium point E^* which is globally asymptotically stable in S, E, I_1, I_2, Q, R

CHAPTER FOUR

RESULTS AND DISCUSSION

4.1 Introduction

In this chapter presents the research results and discussion. Through numerical simulation we depict the detailed information on the dynamics of the drug users in the considered society and the interpretation of the same in real life.

4.2 Numerical Analysis

The section below presents parameter values, numerical analysis and simulation of the drug abuse model, it shows the dynamical behaviour of the drug abuse in the society over the particular period of time.

4.2.1 Parameter Values

The values of the parameters used for numerical simulation in the drug abuse model are presented in Table 2. Since studies on modeling drug abuse are few, the values of parameters used in this study are obtained from few existing related articles and reports and some through estimation using Maximum Likelihood Estimation method, sensitivity analysis and simulations.

Table 2: Parameters values for Drug Abuse Model

Parameters	Value	Source
λ	5	Tang <i>et al.</i> (2021)
ψ_1	0.0001	Estimated
ψ_2	0.0003	Estimated
α_1	0.039	Glantz and Chambers (2006)
ρ	0.2	Njagarah and Nyabadza (2013)
σ_1	0.0301	Orwa and Nyabadza (2019)
ρ_1	0.4	Bhunu and Mushayabasa (2012)
μ_1	0.02	Bradshaw and Timaeus (2006)
ρ_2	0.3	Bhunu and Mushayabasa (2012)
ρ_3	0.3	Orwa and Nyabadza (2019)
α_2	0.0051	Bhunu and Mushayabasa (2012)
α_3	0.0089	Orwa and Nyabadza (2019)
μ_2	0.059	Oyefeso <i>et al.</i> (1999)

4.3 Simulation Results

Figure 2 shows the dynamics of human population when there is drug abuse in the community. Depict the dynamical behavior of the population when no control effort is applied to reduce or eliminate the problem. The simulation shows the significant increase of the drug users both light and heavy at a very high rate. The massive growth of drug users proportionally decrease the susceptible individuals as most of them will be exposed to the environment that may influence drug abuse through adequate contact with the drug users. As assumed adequate and/or frequent contact between the susceptible and drug user may influence the susceptible to start using drug. That is to say the increase of the drug users will increase the rate of contact between the class S and the drug users and thus increase the number of drug users. This is justified by the result in Fig. 2 in which due to the increase of the rate of drug influence the

number of human exposed to disease also increase at a very high rate.

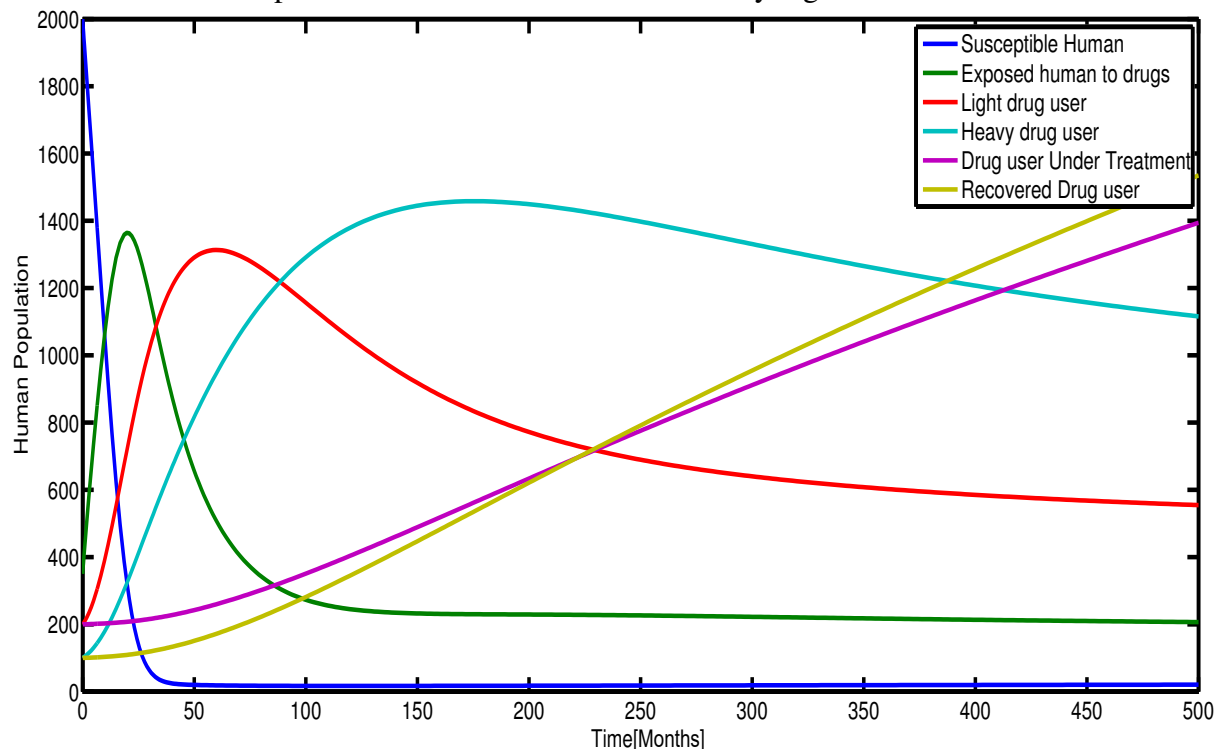


Figure 2: Behavioral Dynamics of Drug Abuse in Human Population

Although the control is not taken seriously as there are no serious effort applied to control the drug abuse apart from the establishment of rehabilitation centers. When the number of drug user increase the number of individual who will be taken to the rehabilitation centers and eventually recover will also increase significantly. Figure 2 shows the increase of the number of people under treatment and those who recover through changing mind or after treatment. As the problem gets serious through natural mechanisms society will eventually find ways to rescue ad hence justify the increase of individual in class Q and R .

Depending on the environment and the individual's resistance to drug addiction, after a certain period of time the light drug user become heavy drug user. This occurs when a light drug user is not taken for treatment in the incubation centers timely. This is to say when there is no efficiency control mechanism the number of heavy drug user increases significantly. Figure 2 also shows the high rate of increase of heavy drug user over time as most of the light drug users will graduate and become heavy drug users due to delayed treatment in the incubation centers.

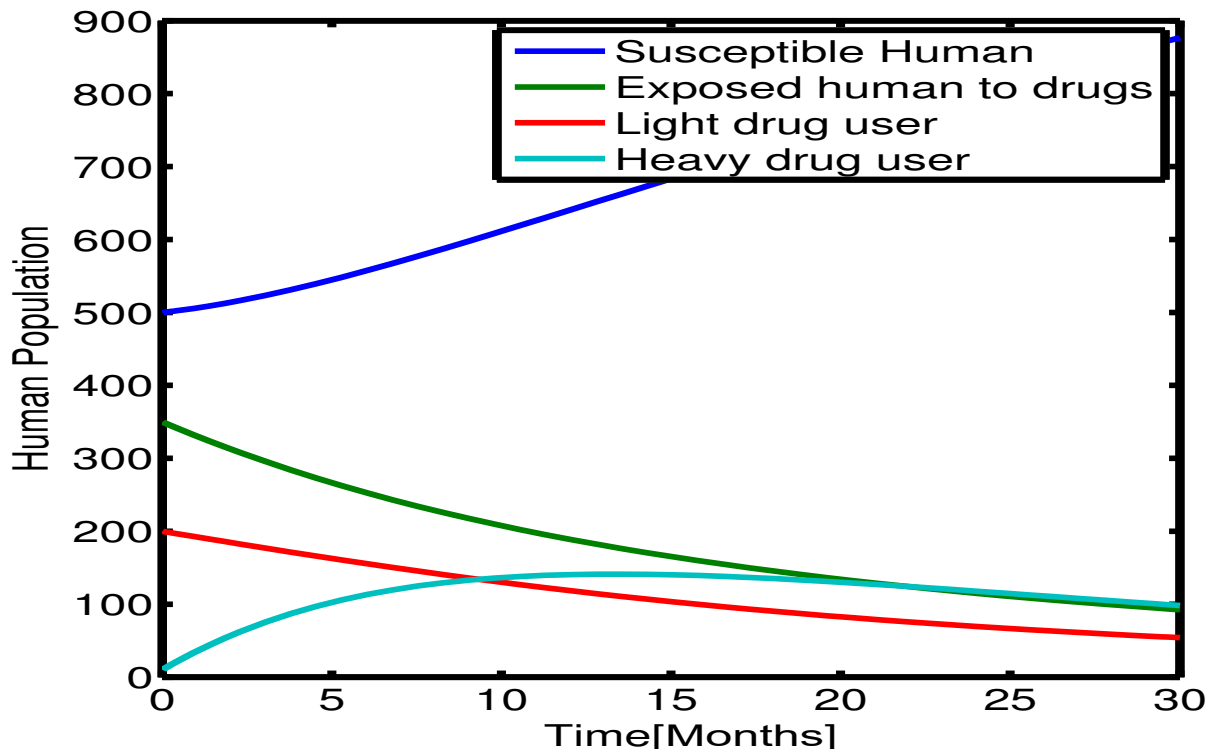


Figure 3: Effect of control Method

The increase of drug abuse in many society is mostly due to delayed treatment that allow the interaction between the drug user and the susceptible individual. The delayed treatment also lead to an increase of heavy drug users that as a result increase the number of death due to drug abuse. This alert the need to apply control mechanism that would reduce the effect of delayed treatment. As a solution to the problem we apply the the early identification and quarantine method as a control method that identifies the drug user earlier and quarantine them for treatment to reduce their influence to the society and protect them from becoming heavy drug addict (see drug abuse model system 1). Figure 3 shows the significant increase of the susceptible human being and a substantial decrease of light and heavy drug users. This dynamical behavior is due to the application of control method that prevent the drug use behavior to affect the susceptible population.

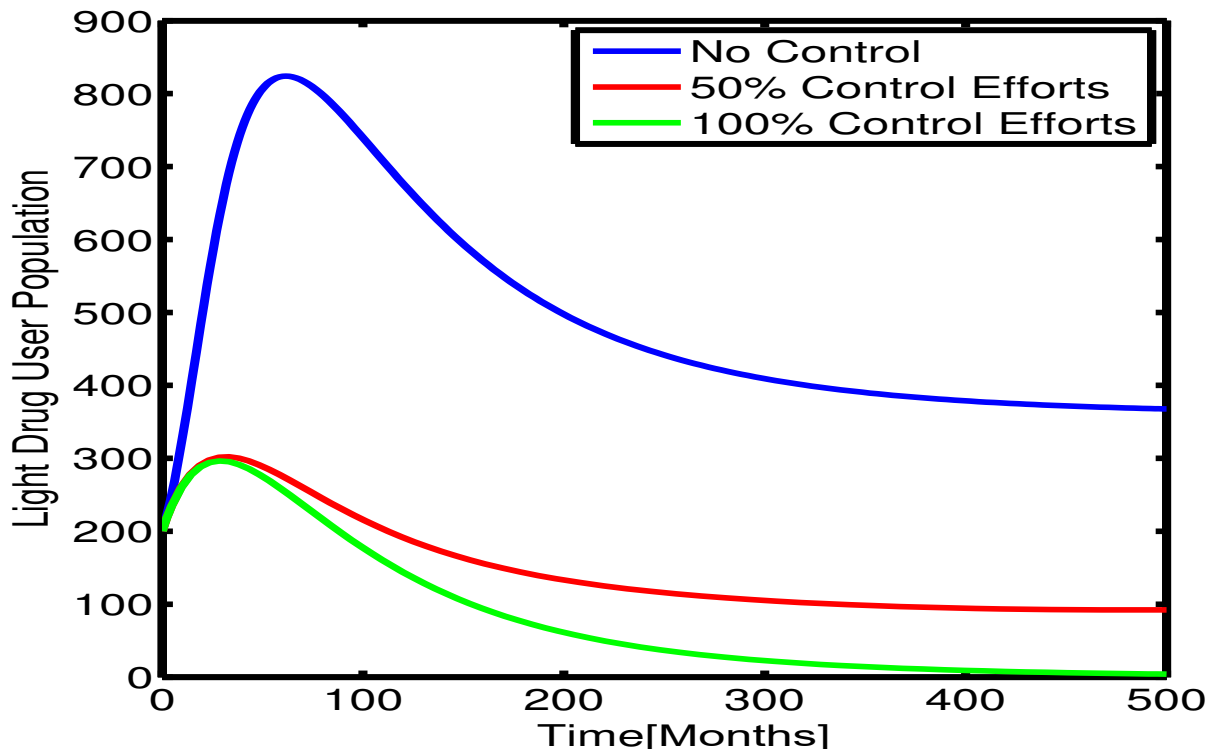


Figure 4: Effect of Variation of level of control in Light Drug Users

The result of any control method depend on the level of effort invested in it, when the effort is high the control results also become higher otherwise the control results will be low. The level of the control method is measured by the efficiency of the control method, the effective point and time the control is applied and the length of the control period. Figure 4 and Fig. 5 shows the effect of different level of effort applied to control drug abuse in the community.

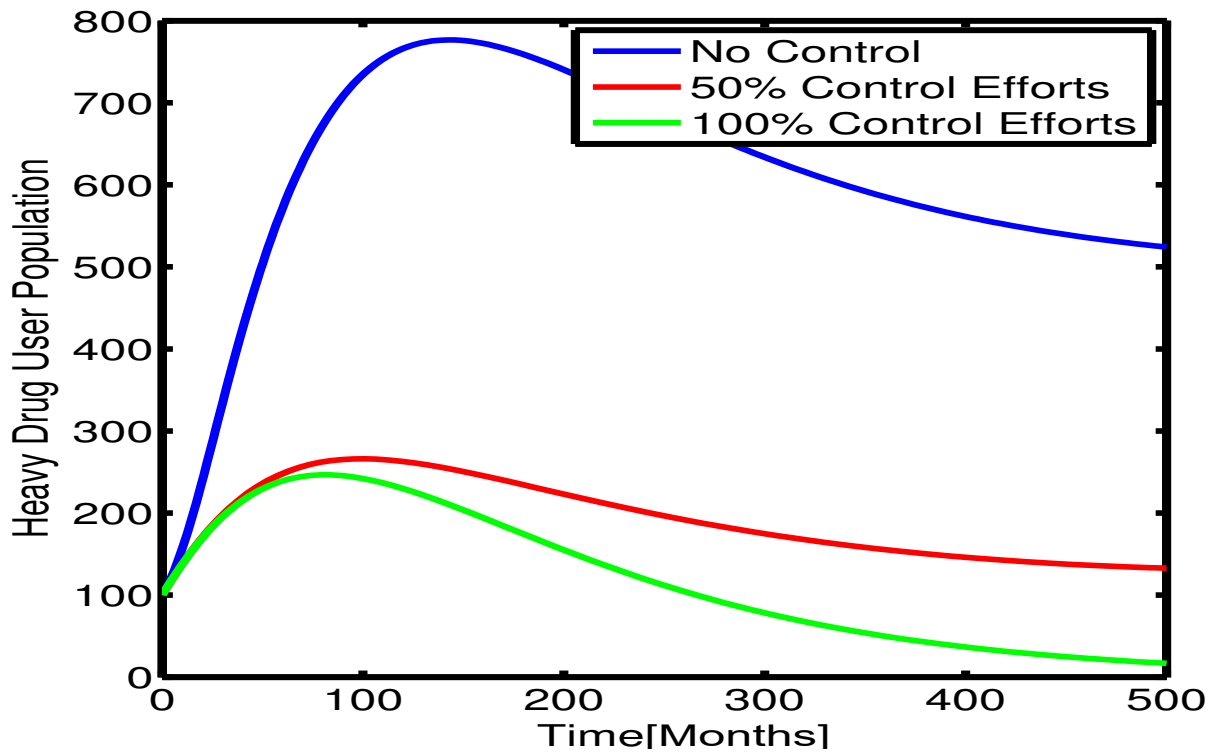


Figure 5: Effect of Variation of level of control in Heavy Drug Users

The result in the two figures shows the significant reduction of the drug users when the control effort is high, but it also gets weaker when the control efforts is reduced. In a case of drug abuse, the positive control results is the result of the timely application of the control method that hinder secondary cases from happening. It deter the rate at which the drug user population influence the drug using behavior to the susceptible population.

CHAPTER FIVE

CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

In this study, the mathematical model to study the dynamical behavior of the community with drug abuse is developed and analyzed. Formulate and analyze the model by establishing the basic and significant condition for the model. Firstly identify through visibility study the factors that dictate the whole dynamics of drug using population in the society. Thus use the identified factors to develop the drug use model in the society using the differential equations. The model divides the human population in six classes which are susceptible (S), Exposed (E), Light drug users (I_1), heavy drug uses (I_2), drug users under treatment (Q) and the recovered drug user (R).

The analytical result shows that the drug abuse model is mathematically meaningful and defined in the positive region Λ . Establish conditions for existence of the equilibrium state and found both drug free and drug use endemic equilibrium point to be locally and asymptotically stable when $R_0 < 1$ and unstable otherwise. Using Metziler matrix and Lypunov function we also established the condition for global stability of drug free and drug use endemic equilibrium points respectively.

Using numerical simulation results shows the dynamical behavior of the population in their classes in the community with drug abuse. The result shows the increase in the drug user (both light and heavy) and the exposed population when the there is no significant control applied to reduce the problem. By default the increase of the number of drug users proportionally increase the number of people under treatment and the recovered drug uses. The result changes when the control is applied to the drug use community. We model the control method namely early identification and quarantine method which identifies the drug users on their early days and take them to quarantine for treatment. The numerical simulation show the significant decrease of the rate at which new drug users are influenced and thus reduce number of drug users in the community which then increase the number of susceptible human who in a way are not risky of becoming drug users.

Numerical result further indicate that, the effective result of the control method depend on the effort applied. When the effort is high the chance to defeat the problem gets higher. The

control method reduce the progression rate of the light drug users to the heavy drug uses. The absence and/or decrease of heavy drug uses reduce the rate of transfer of the drug use behavior to the susceptible human being. Also the small number of heavy drug user reduce the drug use induced death rate significantly.

5.2 Recommendations

The delayed treatment is the most cause of the damages due to drug abuse in many societies. This is due to the fact that it lead to the increase of the number of drug users and as a consequences increases transmission and spread of the drug abuse in the society. Using the result in this study we recommend the following three approaches which when implemented effectively will be a door way to the drug free society.

- (i) The provision of education to the society on the drugs and their effect to human health and the society at large through awareness campaign. The awareness campaign should also equip members of the society with the techniques to identify the drug user earlier before they become hazardous to the society.
- (ii) The proper and timely application of the effective control method that should take into account the most sensitive parameter(s) that lead to the increase of the drug users in the society. The method should also be able to identify where and when to apply the control method in order to get the desired output.
- (iii) Establishment of law and bylaw that would prevent the drugs from spreading to the societies. The stake holders should make sure the people who are engaged in the drug business are punished and/or eliminated from the society to protect the majority.

It is beyond doubt that the involvement of society is very vital to the successful control of drug abuse in the society. It helps to get an holistic understanding of the nature and practice of the drug users and mostly people involved in the drug abuse business. When the society is fully involved it will simplify the identification of the users and the whatever is behind the scenes that would be used to stop or reduce the supply drugs into the society by the appropriate authority.

Future works, beside an excellent work done in this study to study the dynamical behavior of drug abuse in the society but still it does not exhaust 100% of every factor involved in the drug abuse in the society. The study may be adjusted in many ways to produce different results that

would in many ways improve the understanding and control of drug abuse in the society. This study can be extended and adjusted in various ways as given below:

- (i) Including age patterns in modeling the drug abuse in the society.
- (ii) Model the optimal control of the drug abuse model developed in this study.
- (iii) Include the effect of immigrants who are already drug user from the neighboring societies.
- (iv) Include other ways that may lead or influence drug abuse behavior in the society apart from the one considered in this study.

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APPENDICES : Matlab Code

```
function dy = main_ty(t,y)
dy=zeros(size(y))
lambda=5; alpha3=0.00089; psi1=0.0001;psi2=0.0003;mu1=0.00001;rho=0.2;
alpha1=0.039;
rho1=0.4;rho2=0;rho3=0.00011;sigma1=0.0301;sigma2=0.0099991;
mu2=0.001;alpha2=0.0051;
omega1=0;omega2=0;
S=y(1);
E=y(2);
I1=y(3);
I2=y(4);
Q=y(4);
R=y(4);
N=S+E+I1+I2+Q+R;

dy(1)=lambda+alpha3*R-(psi1*I1+psi2*I2+mu1)*(1-omega1)*S;
dy(2)=(psi1*I1+psi2*I2)*(1-omega1)*S-((1-rho)+rho)*alpha1*E-mu1*E;
dy(3)=(1-rho)*alpha1*E-(rho1*(1-
omega2)*sigma1+rho2*sigma1+rho3*sigma1+mu1)*I1;
dy(4)=rho*alpha1*E+rho1*sigma1*(1-omega2)*I1-(sigma2+mu1+mu2)*I2;
dy(5)=rho2*sigma1*I1+sigma2*I2-(alpha2+mu1)*Q;
dy(6)=rho3*sigma1*I1+alpha2*Q-(mu1+alpha3)*R;
```

Execution file

```
clear
tspan=[0 700]
t1span=[0 700]
t2span=[0 700]
y0=[1000, 350, 200, 100, 200, 100]
y10=[1000, 350, 200, 100, 200, 100]
y20=[1000, 350, 200, 100, 200, 100]
[t,y]=ode45(@main_ty,tspan,y0)
[t1,y1]=ode45(@main_ty1,tspan,y0)
[t2,y2]=ode45(@main_ty2,tspan,y0)
%tspan=[0 400];
%y0=[1000, 350, 200, 100, 200, 100];
%[t,y]=ode45(@main_ty,tspan,y0);
figure(1)
plot(t,y(:,4),'Color','B')
hold on
plot(t1,y1(:,4),'Color','R')
hold on
plot(t2,y2(:,4),'Color','G')
%plot(t,y(:,1),t,y(:,2),t,y(:,3),t,y(:,4),t,y(:,5),t,y(:,6))
hold off
%legend('Susceptible Human','Exposed human to drugs','Light drug user','Heavy
drug user', 'Drug user Under Treatment','Recovered Drug user')
legend('Susceptible Human','Exposed human to drugs','Light drug user')
xlabel('Time[Months]')
ylabel('Human Population')
figure(2)
plot(t,y(:,4),'Color','B')
hold on
```

Execution file

```
plot(t1,y1(:,4),'Color','R')
hold on
plot(t2,y2(:,4),'Color','G')
%plot(t,y(:,1),t,y(:,2),t,y(:,3),t,y(:,4),t,y(:,5),t,y(:,6))
hold off
%legend('Susceptible Human','Exposed human to drugs','Light drug user','Heavy
drug user','Drug user Under Treatment','Recovered Drug user')
legend('Susceptible Human','Exposed human to drugs','Light drug user')
xlabel('Time[Months]')
ylabel('Human Population')

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

clear
tspan=[0 1500];
y0=[500, 150, 100, 500];
y10=[500, 150, 100, 500];
y20=[500, 150, 100, 500];
[t,y]=ode45(@main_ty,tspan,y0);
[t,y1]=ode45(@main_ty1,tspan,y0);
[t,y2]=ode45(@main_ty2,tspan,y0);
figure(2)
plot(linspace(0,1500,length(y(:,4))),y(:,4))
hold on
plot(linspace(0,1500,length(y1(:,4))),y1(:,4))
```

Execution file

```
hold on
plot(linspace(0,1500,length(y2(:,4))),y2(:,4))
hold off
%legend('Bacteria in the Environment')
xlabel('Salmonella')
ylabel('Infected Human')
```


Mathematical Modelling of Drug Abuse and it's Effect in the Society

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Abstract. Drug abuse remains to be the global burden causing a large number of death and disability, it is now termed as a significant threat to public health for both developed and developing countries. This work presents a mathematical model as a new approach towards understanding and controlling drug abuse in Tanzania. Using the next-generation matrix method an epidemic threshold value, R_0 is computed and used to establish condition for the existence and stability of stationary points. Using numerical simulation the dynamical behavior of the model is explored and the result shows the significant contribution by the rate of adequate contact between the susceptible individual and the drug user, and the rate of recovery of drug user after rehabilitation as the major factors or players that dictate the dynamics of the drug user population in the society. The model is finally analyzed to study the dynamical behavior of the drug abuse when control is applied, the result shows the drug users are minimized significantly. The result signifies the importance of early control of the drug abuse problem through the establishment of strict laws that will narrow the possibility of this bad practice in societies.

Keywords: Drugs, drug abuse, mathematical model, basic reproductive number, equilibrium points

AMS Mathematics Subject Classification (2010): 92B05

1. Introduction

Drug(s) stand for any chemical substance that when used without the directive of the physician affects a living organism. These substance(s) have got a variety of use but mostly may be used for the treatment of various health problems of a human being [2,12]. The term drug abuse may be defined as the unlawful, illegitimate, prohibited, illegal, inappropriate, extreme or excessive use of drugs. Teens are increasingly engaging in prescribed to relieve several pain and stimulant medication which treats condition like attention deficit disorder and narcolepsy World health organization (2011).

The most common drugs that are used for drug abused includes alcohol, amphetamines, nicotine, alcohol, barbiturates, Cannabis, cocaine, methaqualone, opium

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alkaloids, synthetic opioids and other kind. Drug abuse if are not controlled at the earliest stage may lead to significant effects on society and to the well-being of the user [12]. The excessive use of the drug may result in addiction which may also be termed as dependence or addiction of a particular drug. Drug addiction affects the psychological, social, and physical well-being of the human being and the injurious effect on the whole community[15].

Drug users, use different ways to administer drugs depending on the nature of the drugs one is using. The most popular ways of drug administration include snorting, smoking, and oral ingestion. The mode in which the drug is being administered determines the level of drug uptake in the blood and the related consequences [17].

Africa can not separate itself from the whole world, this means since other continents are affected by the drug abuse problem Africa is also affected significantly. Different studies justify the use of some drugs like alcohol with poverty, this positions Africa onto a more vulnerable situation to be affected by drug abuse problems even more than other developed continents. Many countries in Africa like South Africa and Nigeria have portrayed a worse senatorial as the level of drug use in these countries is almost the peak of drug abuse in Africa and many other countries around the world [13].

Tanzania is also among the African countries that are affected by drug abuse. The problem has to lead to a number of effects to the society in particular youth who are the main workers in different Tanzania's society. Tanzania recorded a high peak of drug trafficking in the years between 1990 and 1995. The government has taken different measures like severe punishment to the drug dealers in order to eradicate the problem but still, the number of drug users is increasing in societies.

The nation's drug report shows the seasonal trend of drug use from 2001 to 2005. In 2005 Tanzania recorded a very significant drop of the use of there was a decrease of use of cocaine of about 1.5 kg per year compared to the 7.5kg per year recorded in 2001 [7]

It is evident that the presence of drug use has attracted many researchers to investigate and search and/or suggest solutions for the problem. Regardless of the extensive work about the same but still the problem still persists and the major factors influencing its existence are still unknown. This justifies the need for a study that will thoroughly study the dynamics of drug abuse in society and identify the factors that dictate its dynamics for a proper control plan.

This study intends to formulate a mathematical model system and analyze it to study the dynamics of drug abuse in society and suggest the control mechanism. The model is analyzed to establish conditions for the existence and extension of the drug abuse problem in the society, the local and global stability of stationary points and the system's dynamical behavior through numerical simulation.

2. Model development

2.1. Model description

In this drug use model, we consider drug user population dynamics in the society, we divide the human beings into six subgroups: first are the human being who have not yet started using drugs but may start to use drugs if they interact with I_1 or I_2 known as susceptible and symbolized by S , second are the people whose company and environment is risky, these are the people who have started using the drugs but they haven't declared or shown symptoms to be termed as light or heavily addicted users to be referred as exposed and

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denoted by E . Third and fourth are the people who are light and heavily addicted users who have shown symptoms and capable of influencing another susceptible individual to start using drugs to be referred to as Light drug users and heavy drug users and denoted by I_1 and I_2 respectively, fifth is the drug users who have started treatment in the rehabilitation centers to be referred the o patient and denoted by Q and the last subgroup are the individuals if treated orthrough self-reflectiononn may recover and move to compartment R where by natural practice they are assumed to be temporary invulnerability from drug use before they become susceptible again.

2.2. Description of interaction

The close relationship and/or frequency contact of susceptible human and the drug users I_1 and I_2 , increase the chances of becoming a drag user and become at a high risky subgroup E at a rate ψ_1 and ψ_2 respectively. Extensive exposure to vulnerable drug use environment leads to light drug user I_1 or heavy drug user I_2 at a rate α_1 and proportional ρ or $(1 - \rho)$ respectively. With time the light drug user may become addicted to drugs become heavy drug user at a rate $\rho_1\sigma_1$. With self-reflection or by changing environment and company light drug addicts may recover and become temporally invulnerable to drug use at a rate $\rho_3\sigma_1$, otherwise depending on the nature of the family and the environment they will be taken to the rehabilitation centers for treatment at the rate $\rho_2\sigma_1$. The heavy drug addicts may be taken to the rehabilitation centers for treatment at a rate σ_2 otherwise if not treated they die due to extensive drug addiction at a rate μ_2 . Successful treatment of drug user in the rehabilitation centers lead to recovery at a rate α_2 , the recovered group become temporally invulnerable to the drug use environment but with time they become susceptible again at a rate α_3 . Human beings are recruited at a constant rate λ and die naturally at the rate μ_1 . ω is the coefficient that represents a control effort using the method of early identification and quarantine of the drug for treatment to reduce the rate of influence that exposes people in the community to drug abuse.

2.3. Parameters description

Parameter	Description
λ	The rate of recruitment of human population
ψ_1 & ψ_2	The rate that S become E due to I_1 and I_2 respectively
α_1	Progression rate out of E to drug user states.
ρ	Proportional of E becoming I_2
σ_1	Rate at which individual leave I_1
ρ_1	Proportional of I_1 becoming I_2
ρ_2	Proportional of I_1 becoming Q
ρ_3	Proportional of I_1 becoming R
α_2	The rate at which Q become R
α_2	The rate at which R become S
μ_1	Natural death rate of human being.
μ_2	Drug addiction induced death rate of human being.
ω	Control efforts to protect S
N	Total human population.

Table 1: Parameters description for Drug use model

2.4. Model assumption

The drug abuse model is developed by using the assumption listed below:

- Human beings in subgroup S are enrolled at a constant rate.
- Same natural death rate for all individuals involved in this study.
- All individuals involved in this study have an equal chance of becoming drug users.
- On recovery individuals become temporarily invulnerable to drug abuse.
- All individuals are born susceptible.
- Only heavy drug user may die due to drug addiction.
- Susceptible individuals become drug users through the social contact process only.

Using the description of interaction and assumption, we summarize the dynamics of drug use in the communities in a compartmental diagram presented in Figure 2.4. The figure captures the interaction between the susceptible, drug users and individuals under treatment.

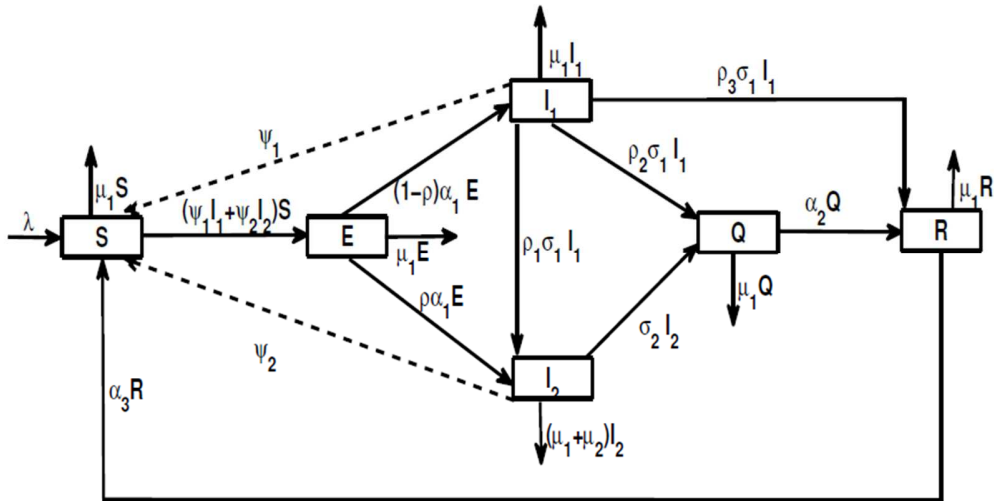


Figure 1: Compartmental model for Drug Use

2.5. Model equation

The dynamics of drug use in the society is presented by system (2.5)

$$\frac{dS}{dt} = \lambda + \alpha_3 R - (\psi_1 I_1 + \psi_2 I_2)(1 - \omega)S - \mu_1 S \quad (1)$$

$$\frac{dE}{dt} = (\psi_1 I_1 + \psi_2 I_2)(1 - \omega)S - ((1 - \rho) + \rho)\alpha_1 E - \mu_1 E \quad (2)$$

$$\frac{dI_1}{dt} = (1 - \rho)\alpha_1 E - (\rho_1 \sigma_1 + \rho_2 \sigma_1 + \rho_3 \sigma_1 + \mu_1)I_1 \quad (3)$$

$$\frac{dI_2}{dt} = \rho\alpha_1 E + \rho_1 \sigma_1 I_1 - (\sigma_2 + \mu_1 + \mu_2)I_2 \quad (4)$$

$$\frac{dQ}{dt} = \rho_2 \sigma_1 I_1 + \sigma_2 I_2 - (\alpha_2 + \mu_1)Q \quad (5)$$

$$\frac{dR}{dt} = \rho_3 \sigma_1 I_1 + \alpha_2 Q - (\mu_1 + \alpha_3)R \quad (6)$$

$$\rho_1 + \rho_2 + \rho_3 = 1$$

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3. Analysis of the model

Here we analyze the basic drug abuse model (without control $\omega = 0$) is presented. Here the mathematical meaningfulness, existence of stationary point and their stability are established. The basic reproduction number R_0 and effective reproduction number R_e are computed using next generation matrix method. We use R_0 to define the condition for global and local stability of the stationary points. In the numerical simulation section we depict the dynamical behavior of the drug abuse model with and without control that aim at reducing the effect of drug abuse in the society.

Therefore in this section we consider the drug abuse model when $\omega = 0$ as in the system 3 below;

$$\frac{dS}{dt} = \lambda + \alpha_3 R - (\psi_1 I_1 + \psi_2 I_2 + \mu_1) S \quad (7)$$

$$\frac{dE}{dt} = (\psi_1 I_1 + \psi_2 I_2) S - ((1 - \rho) + \rho) \alpha_1 E - \mu_1 E \quad (8)$$

$$\frac{dI_1}{dt} = (1 - \rho) \alpha_1 E - (\rho_1 \sigma_1 + \rho_2 \sigma_1 + \rho_3 \sigma_1 + \mu_1) I_1 \quad (9)$$

$$\frac{dI_2}{dt} = \rho \alpha_1 E + \rho_1 \sigma_1 I_1 - (\sigma_2 + \mu_1 + \mu_2) I_2 \quad (10)$$

$$\frac{dQ}{dt} = \rho_2 \sigma_1 I_1 + \sigma_2 I_2 - (\alpha_2 + \mu_1) Q \quad (11)$$

$$\frac{dR}{dt} = \rho_3 \sigma_1 I_1 + \alpha_2 Q - (\mu_1 + \alpha_3) R \quad (12)$$

4. Basic properties of the model

4.1. Invariant region

Due to the fact that drugs affect human population, then, in order to model this process we need to assume that all of the defined state variables and parameters used in the model are non-negative for $\forall t \geq 0$. We are required to analyze the drug abuse model so that it is defined in a suitable feasible region which has positive state variables and therefore result to Theorem 4.1; The existing forward solutions in R_+^6 of the drug abuse model system are feasible $\forall t \geq 0$ whenever they enter the invariant region namely Λ . where

$$\Lambda = (S, E, I_1, I_2, Q, R) \in R_+^6: S + E + I_1 + I_2 + Q + R \leq N$$

and Λ is the positive invariant region of drug abuse model system

4.2. Positivity of the solution

Here we need to show that variables and parameters used in the drug abuse model are non negative $\forall t \geq 0$. Assume the initial values of the drug abuse model system (3) to be: $(S(0) > 0$ and $(E(0), I_1(0), I_2(0), Q(0), R(0)) \geq 0$). Then the model's solution set $S(t), E(t), Q(t), I_1(t), I_2(t)$ and $R(t)$ are positive $\forall t \geq 0$.

5. Analysis of the stationary points and R_0

This part of the study present the existence of stationary states, the number of drug use cases produced by one drug user in the whole drug user period and stability of the stationary points.

5.1. Drug use free equilibrium

We obtain the Drug use free equilibrium by setting $E = Q = I_1 = I_2$ and $R = 0$ and

substitute to the drug abuse model system (3).

The drug use free-equilibrium point of the drug abuse model system is as given in (13)

$$X_0(S^0, E^0, I_1^0, I_2^0, Q^0, R^0) = \left(\frac{\lambda}{\mu_1}, 0, 0, 0, 0, 0 \right). \quad (13)$$

5.2. Basic reproduction number R_0 for drug abuse

This is the number of secondary drug abuse cases that are to be produced by one drug user in the whole drug abuse period of that particular individual in a population defined by only susceptible population. The criteria for this dimensionless parameter is that if $R_0 < 1$, then the single drug user in a population defined by only susceptible population may influence less than one individual to start using drugs. This entails that drug abuse problem may be eradicated from the population and the drug free stationary point is asymptotically stable which also means that the drug abuse cannot attack the society.

When $R_0 > 1$ it portray that one drug user in a population defined by only susceptible population may influence more than one individuals to start using drugs. This also means that drug abuse problem may Hence the drug abuse may continue to stay in the society. This situation also means that the drug free equilibrium point is unstable and that it is vividly clear that the drug abuse problem can attack the society and stay for a long time.

And if $R_0 = 1$ it portray that one drug user in entirely susceptible population influence drug use to one new human being. Hence the drug abuse problem will be alive in the society without an serious epidemic as narrated by [1].

To find the basic reproduction number we use next generation method by [19]. Consider a heterogeneous population in compartments S, E, I_1, I_2, Q and R arranged such that m drug using classes come first.

Assume $F_i(x)$ as rate of entrance of new drug users in class i , $V_i^+(x)$ rate of transfer of individuals in the class i by any other means except the drug abuse induced $V_i^-(x)$ be the rate of transfer of individuals out of class i .

The the model system is as presented below;

$$x_i' = F_i(x) - V_i(x)$$

where $V_i(x) = V_i^-(x) - V_i^+(x)$.

Then we use x_0 , to find the $m \times m$ matrices F and V

$$F = \left(\frac{\partial F_i}{\partial x_j}(x_0) \right), V = \left(\frac{\partial V_i}{\partial x_j}(x_0) \right)$$

with $1 \leq i, j \leq m$.

By using the study by [5] we call Matrix FV^{-1} , a next generation matrix Thus:

$$R_0 = \rho(FV^{-1})$$

$$\frac{dE}{dt} = (\psi_1 I_1 + \psi_2 I_2)S - ((1 - \rho) + \rho)\alpha_1 E - \mu_1 E \quad (14)$$

$$\frac{dI_1}{dt} = (1 - \rho)\alpha_1 E - (\rho_1 \sigma_1 + \rho_1 \sigma_1 + \rho_1 \sigma_1 + \mu_1)I_1 \quad (15)$$

$$\frac{dI_2}{dt} = \rho E + \rho_1 \sigma_1 I_1 - (\sigma_2 + \mu_1 + \mu_2)I_2 \quad (16)$$

$$\frac{dQ}{dt} = \rho_2 \sigma_1 I_1 + \sigma_2 I_2 - (\alpha_2 + \mu_1)Q \quad (17)$$

$$\frac{dR}{dt} = \rho_3 \sigma_1 I_1 + \alpha_2 Q - (\mu_1 + \alpha_3)R \quad (18)$$

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$$\frac{dS}{dt} = \lambda + \alpha_3 R - (\psi_1 I_1 + \psi_2 I_2 + \mu_1) S \quad (19)$$

Now from the system (5.2) the drug using classes are the one with compartment E, I_1, I_2 and Q , this will now yield

$$\mathbf{F}_i = \begin{pmatrix} (\psi_1 I_1 + \psi_2 I_2) S \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (20)$$

And

$$\mathbf{V}_i = \begin{pmatrix} \alpha_1 E + \mu_1 E \\ (\rho_1 \sigma_1 + \mu_1) I_1 - (1 - \rho) \alpha_1 E \\ (\sigma_2 + \mu_1 + \mu_2) I_2 - \rho E - \rho_1 \sigma_1 I_1 \\ (\alpha_2 + \mu_1) Q - \rho_2 \sigma_1 I_1 - \sigma_2 I_2 \end{pmatrix}. \quad (21)$$

The computation for matrices F and V at x_0 are as given below;

$$\frac{\partial F_i}{\partial x_j} = \begin{pmatrix} \frac{\partial F_1}{\partial E} & \frac{\partial F_1}{\partial I_1} & \frac{\partial F_1}{\partial I_2} & \frac{\partial F_1}{\partial Q} \\ \frac{\partial F_2}{\partial E} & \frac{\partial F_2}{\partial I_1} & \frac{\partial F_2}{\partial I_2} & \frac{\partial F_2}{\partial Q} \\ \frac{\partial F_3}{\partial E} & \frac{\partial F_3}{\partial I_1} & \frac{\partial F_3}{\partial I_2} & \frac{\partial F_3}{\partial Q} \\ \frac{\partial F_4}{\partial E} & \frac{\partial F_4}{\partial I_1} & \frac{\partial F_4}{\partial I_2} & \frac{\partial F_4}{\partial Q} \end{pmatrix} = \begin{pmatrix} 0 & \psi_1 S & \psi_2 S & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now at x_0 we will have

$$\mathbf{F} = \begin{pmatrix} 0 & \psi_1 \frac{\lambda}{\mu_1} & \psi_2 \frac{\lambda}{\mu_1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (22)$$

and for V we will have;

$$V = \frac{\partial V_i}{\partial x_j}(x_0)$$

$$\mathbf{V} = \begin{pmatrix} \alpha_1 + \mu_1 & 0 & 0 & 0 \\ -(1 - \rho) \alpha_1 & (\rho_1 \sigma_1 + \mu_1) & 0 & 0 \\ -\rho \alpha_1 & -\rho_1 \sigma_1 & (\sigma_2 + \mu_1 + \mu_2) & 0 \\ 0 & -\rho_2 \sigma_1 & -\sigma_2 & (\alpha_2 + \mu_1) \end{pmatrix}. \quad (23)$$

We can obtain V^{-1} and FV^{-1} using maple, then the spectral radius which is also the basic reproduction number is as in (24);

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$$R_0 = \frac{((\eta\psi_1 + \rho\psi_2)\mu_1 + \psi_2\rho_1(\eta + \rho)\sigma_1 + \eta\psi_1(\sigma_2 + \mu_2))\alpha_1\lambda}{\mu_1(\alpha_1 + \mu_1)(\sigma_2 + \mu_1 + \mu_2)(\rho_1\sigma_1 + \mu_1)} \quad (24)$$

where $\eta = (1 - \rho)$.

When early identification and quarantine control method is applied, the number of secondary cases of drug users will decrease which as a result will reduce the number of adequate contact between the drug user and the susceptible population. Now using the same next generation matrix method we can generate an effective reproduction number that consider the effect of control as in (25)

$$R_e = \frac{((\eta\psi_1 + \rho\psi_2)\mu_1 + \psi_2\rho_1(\eta + \rho)\sigma_1 + \eta\psi_1(\sigma_2 + \mu_2))(1 - \omega)\alpha_1\lambda}{\mu_1(\alpha_1 + \mu_1)(\sigma_2 + \mu_1 + \mu_2)(\rho_1\sigma_1 + \mu_1)} \quad (25)$$

where $\eta = (1 - \rho)$.

When $\omega \neq 0$ it reduces the value of the number of secondary cases produced as it can be seen in the expression (25), it can be seen that the increase of the control efforts will reduce the value of the expression in (25) which the number of secondary cases of drug users produced by a drug user in the entire period of drug abuse, but when $\omega = 0$ the expression (25) become the basic reproduction number as given in (24).

6. Steady state and local stability of the critical points

This section presents the conditions for stability of critical points of the drug abuse model system (3).

6.1. Disease free equilibrium

The model has a drug free equilibrium in which we set all infectious compartment and the derivatives equal to zero. Then we have the drug free-equilibrium point is as given in (26)

$$X_0(S^0, E^0, I_1^0, I_2^0, Q^0, R^0) = \left(\frac{\lambda}{\mu_1}, 0, 0, 0, 0, 0\right). \quad (26)$$

6.2. Local stability of the drug-free equilibrium point

This section presents the analysis for local stability of the drug free stationary point of the drug abuse model. We use Jacobian method by considering that all equations in drug abuse model in (3) are analyzed at the drug free stationary point X_0 . We are required to compute and asses the eigenvalues of Jacobian matrix in order to verify that the drug free stationary point is locally and asymptotically stable. Further more we need to show that the real parts of the eigenvalues of the matrix at X_0 are negative.

Using the concept by [10], we are required to show that eigenvalues are negative, in which we need to prove that determinant of the Jacobian matrix is positive and its trace negative. .

Now using the Jacobian matrix of the system (3) at X_0 we can prove that, drug free stationary point E^0 is locally asymptotically stable and leads to the following theorem:

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$$\begin{aligned}
 & \mathbf{J}(\mathbf{X}_0) \\
 = & \begin{pmatrix} -\mu_1 & 0 & \frac{\psi_1 \lambda}{\mu_1} & -\frac{\psi_2 \lambda}{\mu_1} & 0 & \alpha_3 \\ 0 & -(\alpha_1 + \mu_1) & \frac{\lambda \psi_1}{\mu_1} & \frac{\psi_2 \lambda}{\mu_1} & 0 & 0 \\ 0 & (1 - \rho)\alpha_1 & -(\sigma_1 + \mu_1) & 0 & 0 & 0 \\ 0 & \rho\alpha_1 & \rho_1 \sigma_1 & -(\sigma_2 + \mu_1 + \mu_2) & 0 & 0 \\ 0 & 0 & \rho_2 \sigma_1 & \sigma_2 & -(\alpha_2 + \mu_1) & 0 \\ 0 & 0 & \rho_3 \sigma_1 & 0 & \alpha_2 & -(\mu_1 + \alpha_3) \end{pmatrix}
 \end{aligned} \tag{27}$$

Using trace and determinant method; we need to check if trace is negative and the determinant of matrix (27) is positive.

Computing we get trace of the matrix (27) as given below;

$$\mathbf{Trace} = -\mu - (\alpha_1 + \mu_1) - (\sigma_1 + \mu_1) - (\sigma_2 + \mu_1 + \mu_2) - (\alpha_2 + \mu_1) - (\omega + \mu) - (\mu_1 + \alpha_3)$$

Therefore the matrix (27) is negative.

We then compute determinant of the matrix (27) in which it is positive if

$$\frac{((\eta\psi_1 + \rho\psi_2)\mu_1 + \psi_2\rho_1(\eta + \rho)\sigma_1 + \eta\psi_1(\sigma_2 + \mu_2))\alpha_1\lambda}{\mu_1(\alpha_1 + \mu_1)(\sigma_2 + \mu_1 + \mu_2)(\rho_1\sigma_1 + \mu_1)} < 1$$

where

$$\frac{((\eta\psi_1 + \rho\psi_2)\mu_1 + \psi_2\rho_1(\eta + \rho)\sigma_1 + \eta\psi_1(\sigma_2 + \mu_2))\alpha_1\lambda}{\mu_1(\alpha_1 + \mu_1)(\sigma_2 + \mu_1 + \mu_2)(\rho_1\sigma_1 + \mu_1)} \tag{28}$$

represents the basic reproduction number, R_0 .

The above results justifies that the drug free stationary point x^0 is locally asymptotically stable as in theorem below: The Drug Free stationary point X_0 is locally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

6.3. Global stability of the drug-free equilibrium point

This section presents the analysis of the global stability of the drug free stationary point. We use the method by [4] known as Metzler matrix as presented in the steps below;

Assume \mathbf{Y}_n to be the vector of compartments which can not influence any drug user, \mathbf{Y}_i to be the vector for classes that can influence individuals to start using drugs and $\mathbf{Y}_{\mathbf{X}_0, n}$ to be the vector of drug free stationary point.

$$\begin{cases} \frac{d\mathbf{Y}_n}{dt} = A_1(\mathbf{Y}_n - \mathbf{Y}_{\mathbf{X}_0, n}) + A_2\mathbf{Y}_i \\ \frac{d\mathbf{Y}_i}{dt} = A_3\mathbf{Y}_i \end{cases} \tag{29}$$

this will then yield;

$$\mathbf{Y}_n = (S, R, E, Q)^T \quad \mathbf{Y}_i = (I_1, I_2) \quad \mathbf{Y}_{\mathbf{X}_0, n} = \left(\frac{\vartheta}{\mu}, 0\right)$$

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$$\mathbf{Y}_n - \mathbf{Y}_{\mathbf{X}_0, n} = \begin{pmatrix} S - \frac{\lambda}{\mu_1} \\ R \\ E \\ Q \end{pmatrix}$$

Using the Metzler matrix method we are required to show that the eigen values of a matrix A_1 are negative and A_3 is a Metzler matrix which represent a matrix with non-negative off diagonal element. The subsystem (29) then becomes;

$$\begin{pmatrix} \lambda + \alpha_3 R - (\psi_1 I_1 + \psi_2 I_2 + \mu_1) S, \\ \rho_3 \sigma_1 I_1 + \alpha_2 Q - (\mu_1 + \alpha_3) R. \\ (\psi_1 I_1 + \psi_2 I_2) S - ((1 - \rho) + \rho) \alpha_1 E - \mu_1 E. \\ \rho_2 \sigma_1 I_1 + \sigma_2 I_2 - (\alpha_2 + \mu_1) Q. \end{pmatrix} = A_1 \begin{pmatrix} S - \frac{\lambda}{\mu} \\ R \\ E \\ Q \end{pmatrix} + A_2 \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

and

$$\begin{pmatrix} (1 - \rho) \alpha_1 E - (\rho_1 \sigma_1 + \rho_2 \sigma_1 + \rho_3 \sigma_1 + \mu_1) I_1, \\ \rho \alpha_1 E + \rho_1 \sigma_1 I_1 - (\sigma_2 + \mu_1 + \mu_2) I_2. \end{pmatrix} = A_3 \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

We then use the drug use influencing and non-influencing element from the general Drug abuse model model to get matrices (30):

$$\mathbf{A}_1 = \begin{pmatrix} -\mu_1 & \alpha_3 & 0 & 0 \\ 0 & -(\mu_1 + \alpha_3) & 0 & \alpha_2 \\ 0 & 0 & -(\alpha_1 + \mu_1) & 0 \\ 0 & 0 & 0 & -(\alpha_2 + \mu_1) \end{pmatrix} \quad (30)$$

$$\mathbf{A}_2 = \begin{pmatrix} -\frac{\lambda \psi_1}{\mu_1} & -\frac{\lambda \psi_2}{\mu_1} \\ \rho_3 \sigma_1 & 0 \\ \frac{\lambda \psi_1}{\mu_1} & \frac{\lambda \psi_2}{\mu_1} \\ \mu_1 & \mu_1 \\ \rho_2 \sigma_1 & \sigma_2 \end{pmatrix} \quad (31)$$

$$\mathbf{A}_3 = \begin{pmatrix} -(\sigma_1 + \mu_1) & 0 \\ \rho_1 \sigma_1 & -(\sigma_2 + \mu_1 + \mu_2) \end{pmatrix} \quad (32)$$

It is clearly seen that eigenvalues values of a matrix A_1 , are real and negative. This justifies that the system

$$\frac{d\mathbf{Y}_n}{dt} = A_1(\mathbf{Y}_n - \mathbf{Y}_{\mathbf{X}_0, n}) + A_2 \mathbf{Y}_i$$

is globally and asymptotically stable at $\mathbf{Y}_{\mathbf{X}_0}$.

We can also see that; A_3 has non-negative off-diagonal elements which means it is a Metzler stable matrix. Therefore Drug Free Equilibrium point for drug abuse model system is globally asymptotically stable as in the theorem below: The drug-free equilibrium point is globally asymptotically stable in E_0 if $R_0 < 1$ and unstable if $R_0 > 1$.

6.4. Existence of drug abuse endemic equilibrium

In this section we establish the conditions for existence of the drug abuse endemic equilibrium point of the system (3).

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To obtain the drug abuse endemic equilibrium point $E^*(S^*, E^*, I_1^*, I_2^*, Q^*, R^*)$ we set system (3) equal to zero and solve equations.

The model system (6.4) which exist for $R_0 > 1$.

$$\lambda + \alpha_3 R - (\psi_1 I_1 + \psi_2 I_2 + \mu_1) S = 0 \quad (33)$$

$$(\psi_1 I_1 + \psi_2 I_2) S - ((1 - \rho) + \rho) \alpha_1 E - \mu_1 E = 0 \quad (34)$$

$$(1 - \rho) \alpha_1 E - (\rho_1 \sigma_1 + \rho_2 \sigma_1 + \rho_3 \sigma_1 + \mu_1) I_1 = 0 \quad (35)$$

$$\rho \alpha_1 E + \rho_1 \sigma_1 I_1 - (\sigma_2 + \mu_1 + \mu_2) I_2 = 0 \quad (36)$$

$$\rho_2 \sigma_1 I_1 + \sigma_2 I_2 - (\alpha_2 + \mu_1) Q = 0 \quad (37)$$

$$\rho_3 \sigma_1 I_1 + \alpha_2 Q - (\mu_1 + \alpha_3) R = 0 \quad (38)$$

Studies by [18] and that of [11] portray how to we prove the existence the drug abuse endemic equilibrium points in which we are required to satisfy the condition $E \neq 0$ or $Q \neq 0$ or $I_1 \neq 0$ or $I_2 \neq 0$ that is $S > 0$ or $E > 0$ or $Q > 0$ or $I_1 > 0$ or $I_2 > 0$ or $R > 0$. Adding the equation in the system (6.4) yields (39)

$$\lambda - \mu_1(S + E + I_1 + I_2 + Q + R) - \mu_2 I_2 = 0 \quad (39)$$

We then have

$$\lambda = \mu_1 N + \mu_2 I_2$$

Therefore since $\lambda > 0$, $\mu_1 > 0$ and $\mu_2 > 0$ we can discern that $\mu_1 N > 0$ and $\mu_2 I_2 > 0$ implying that $S > 0$, $E > 0$, $I_1 > 0$, $I_2 > 0$, $Q > 0$ and $R > 0$.

This prove that the drug abuse endemic equilibrium point of the drug abuse model system exists.

6.5. Global stability of drug abuse endemic equilibrium point

This section analyzes the condition for stability of the drug abuse endemic equilibrium points. The results in the study by [19], shows that the local stability of the Drug Free Equilibrium implies that the local stability of the Drug Abuse Endemic Equilibrium is also locally stable for the reverse condition.

We use Korobeinikov approach as described by [?, ?, ?] to find the global stability of Drug Abuse Endemic stationary point.

We thus formulate a suitable Lyapunov function for drug abuse model system in the following form:

$$V = \sum a_i (y_i - y_i^* \ln y_i)$$

where a_i is an appropriately selected positive constant, y_i is a population of the i^{th} class type, and y_i^* is the stationary point.

We will then have

$$V = W_1(S - S^* \ln S) + W_2(E - E^* \ln E) + W_4(I_2 - I_1^* \ln I_1) + W_5(I_2 - I_2^* \ln I_2) + W_3(Q - Q^* \ln Q) + W_6(R - R^* \ln R)$$

The constants W_i are non-negative in Λ such that $W_i > 0$ for $i = 1, 2, 3, \dots, 6$. V which is the Lyapunov function W_1, W_2, \dots, W_6 defined as the Lyapunov function constant are selected so that V should be continuous and differentiable in a space.

Finding the time derivative of V we get:

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$$\begin{aligned} \frac{dV}{dt} = & W_1(1 - \frac{S^*}{S}) \frac{dS}{dt} + W_2(1 - \frac{E^*}{E}) \frac{dE}{dt} + W_3(1 - \frac{I_1^*}{I_1}) \frac{dI_1}{dt} \\ & + W_4(1 - \frac{I_2^*}{I_2}) \frac{dI_2}{dt} + W_5(1 - \frac{Q^*}{Q}) \frac{dQ}{dt} + W_6(1 - \frac{R^*}{R}) \frac{dR}{dt} \end{aligned}$$

Substituting the respective equations from the drug abuse model system (3) we get;

$$\begin{aligned} \frac{dV}{dt} = & W_1(1 - \frac{S^*}{S})[\lambda + \alpha_3 R - (\psi_1 I_1 + \psi_2 I_2 + \mu_1)S] \\ & + W_2(1 - \frac{E^*}{E})[(\psi_1 I_1 + \psi_2 I_2)S - ((1 - \rho) + \rho)\alpha_1 E - \mu_1 E] \\ & + W_3(1 - \frac{I_1^*}{I_1})[(1 - \rho)\alpha_1 E - (\rho_1 \sigma_1 + \rho_2 \sigma_1 + \rho_3 \sigma_1 + \mu_1)I_1] \\ & + W_4(1 - \frac{I_2^*}{I_2})[\rho \alpha_1 E + \rho_1 \sigma_1 I_1 - (\sigma_2 + \mu_1 + \mu_2)I_2] \\ & + W_5(1 - \frac{Q^*}{Q})[\rho_2 \sigma_1 I_1 + \sigma_2 I_2 - (\alpha_2 + \mu_1)Q] \\ & + W_6(1 - \frac{R^*}{R})[\rho_3 \sigma_1 I_1 + \alpha_2 Q - (\mu_1 + \alpha_3)R] \end{aligned}$$

Computing time derivative of V at drug abuse endemic stationary point yields:

$$\begin{aligned} \frac{dV}{dt} = & -W_1(1 - \frac{S^*}{S})^2 - W_2(1 - \frac{E^*}{E})^2 - W_3(1 - \frac{I_1^*}{I_1})^2 \\ & - W_4(1 - \frac{I_2^*}{I_2})^2 - W_5(1 - \frac{Q^*}{Q})^2 - W_6(1 - \frac{R^*}{R})^2 \\ & + F(S, E, I_1, I_2, Q, R) \end{aligned}$$

where the function $F(S, E, I_1, I_2, Q, R)$ is non positive,

Using the insight from the study by [14] and [8]. We have

$F(S, E, I_1, I_2, Q, R) \leq 0$ for all S, E, I_1, I_2, Q, R , Then $\frac{dV}{dt} \leq 0$ for all S, E, I_1, I_2, Q, R and it is zero when $S = S^*, E = E^*, Q = Q^*, I_1 = I_1^*, I_2 = I_2^*, R = R^*$ Hence the largest compact invariant set in S, E, I_1, I_2, Q, R such that $\frac{dV}{dt} = 0$ is the singleton E^* which is Drug Abuse Endemic stationary point of the model system (3).

By LaSalle's invariant principle in [9] we proclaim that E^* is globally asymptotically stable in the interior of drug abuse model system region of S, E, I_1, I_2, Q, R as in Theorem 6.5

If $R_0 > 1$ then the drug abuse model system (3) has a distinctive drug abuse endemic equilibrium point E^* which is globally asymptotically stable in S, E, I_1, I_2, Q, R

7. Numerical analysis and simulation

The section below presents parameter values, numerical analysis and simulation of the drug abuse model, it shows the dynamical behaviour of the drug abuse in the society over the particular period of time.

7.1. Parameter values

The values of the parameters used for numerical simulation in the drug abuse model are presented in Table 2. The parameters that are obtained from existing related articles and reports and some are estimated.

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Parameters	Value	Source
λ	5	Estimated
ψ_1	0.0001	Estimated
ψ_2	0.0003	Estimated
α_1	0.039	[6]
ρ	0.2	Estimated
σ_1	0.0301	Estimated
ρ_1	0.4	Estimated
μ_1	0.02	[3]
ρ_2	0.3	Estimated
ρ_3	0.3	Estimated
α_2	0.0051	Estimated
α_3	0.0089	Estimated
μ_2	0.059	[16]

Table 2: Parameters values for Drug Abuse model.

7.2. Results and discussion

Figure 7.2 shows the dynamics of human population when there is drug abuse in the community. We depict the dynamical behavior of the population when no control effort is applied to reduce or eliminate the problem. The simulation shows the significant increase of the drug users both light and heavy at a very high rate. The massive growth of drug users proportionally decrease the susceptible individuals as most of them will be exposed to the environment that may influence drug abuse through adequate contact with the drug users. As assumed adequate and/or frequent contact between the susceptible and drug user may influence the susceptible to start using drug. That is to say the increase of the drug users will increase the rate of contact between the class S and the drug users and thus increase the number of drug users. This is justified by the result in Figure 7.2 in which due to the increase of the rate of drug influence the number of human exposed to disease also increase at a very high rate.

Although the control is not taken seriously as there are no serious effort applied to control the drug abuse apart from the establishment of rehabilitation centers. When the number of drug user increase the number of individual who will be taken to the rehabilitation centers and eventually recover will also increase significantly. Figure 7.2 show the increase of the number of people under treatment and those who recover through changing mind or after treatment. As the problem gets serious through natural mechanisms society will eventually find ways to rescue ad hence justify the increase of individual in class Q and R .

Depending on the environment and the individual's resistance to drug addiction, after a certain period of time the light drug user become heavy drug user. This occurs when a light drug user is not taken for treatment in the incubation centers timely. This is to say when there is no efficiency control mechanism the number of heavy drug user increases significantly. Figure 7.2 also shows the high rate of increase of heavy drug user over time as most of the light drug users will graduate and become heavy drug users due to delayed treatment in the incubation centers.

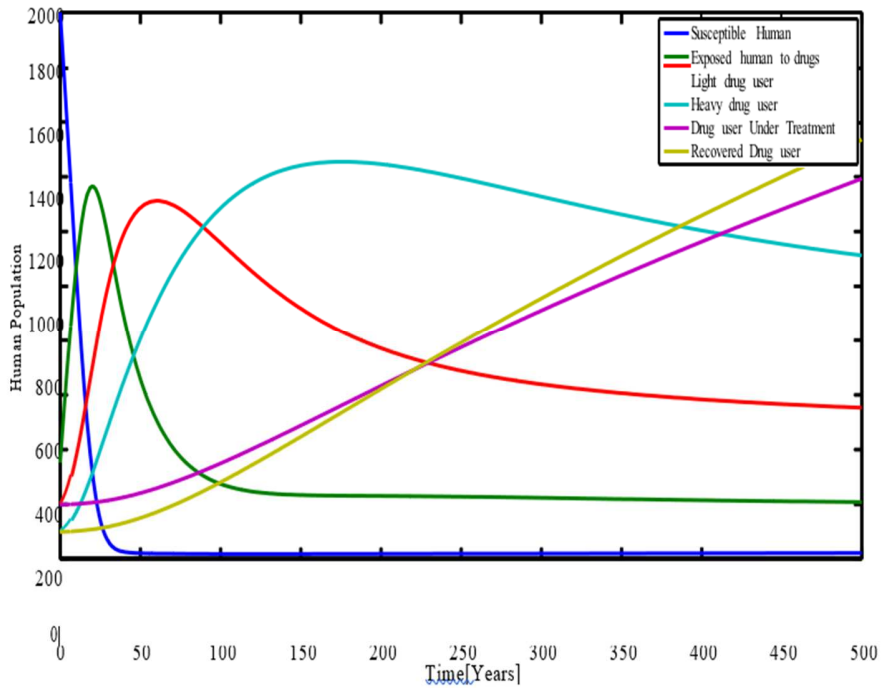


Figure 2: Behavioral Dynamics of Drug Abuse in Human Population

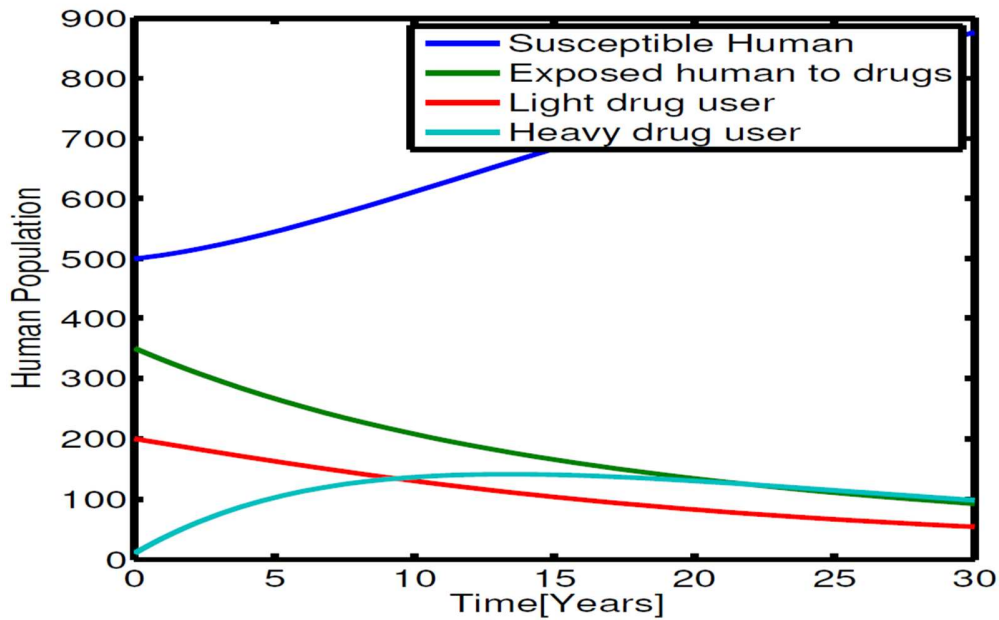


Figure 3: Effect of control Method

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The increase of drug abuse in many society is mostly due to delayed treatment that allow the interaction between the drug user and the susceptible individual. The delayed treatment also lead to an increase of heavy drug users that as a result increase the number of death due to drug abuse. This alert the need to apply control mechanism that would reduce the effect of delayed treatment. As a solution to the problem we apply the the early identification and quarantine method as a control method that identifies the drug user earlier and quarantine them for treatment to reduce their influence to the society and protect them from becoming heavy drug addict (see drug abuse model system 2.5). Figure 7.2 shows the significant increase of the susceptible human being and a substantial decrease of light and heavy drug users. This dynamical behavior is due to the application of control method that prevent the drug use behavior to affect the susceptible population.

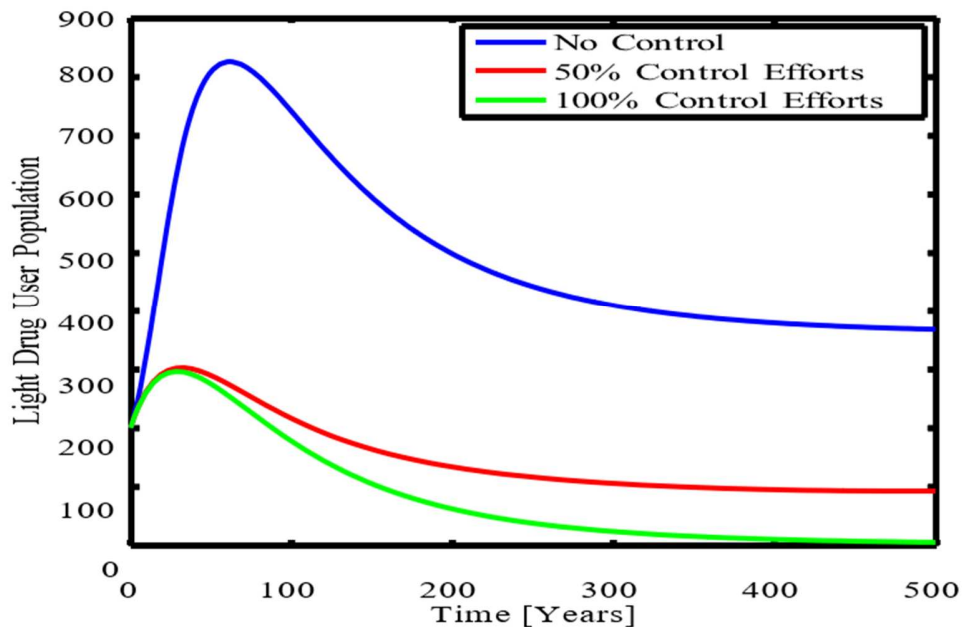


Figure 4: Effect of Variation of level of control in Light Drug Users

The result of any control method depend on the level of effort invested in it, when the effort is high the control results also become higher otherwise the control results will be low. The level of the control method is measured by the efficiency of the control method, the effective point and time the control is applied and the length of the control period. Figure 7.2 and Figure 7.2 shows the effect of different level of effort applied to control drug abuse in the community.

The result in the two figures shows the significant reduction of the drug users when the control effort is high, but it also gets weaker when the control efforts is reduced. In a case of drug abuse, the positive control results is the result of the timely application of the control method that hinder secondary cases from happening. It deter the rate at which the drug user population influence the drug using behavior to the susceptible population.

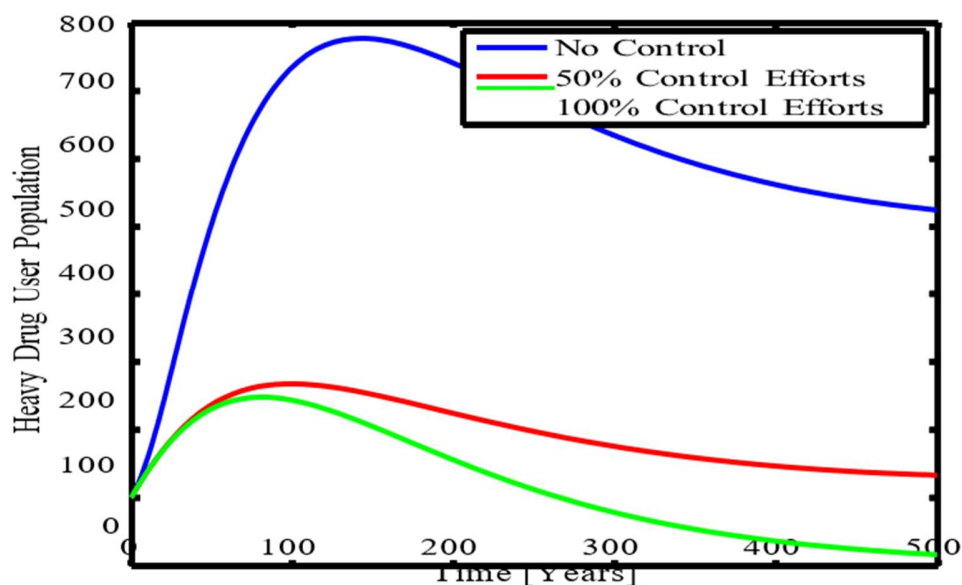


Figure 5: Effect of Variation of level of control in Heavy Drug Users

8. Conclusion

In this study, the mathematical model to study the dynamical behavior of the community with drug abuse is developed and analyzed. We formulate and analyze the model by establishing the basic and significant condition for the model. The analytical result shows that the drug abuse model is mathematically meaningful and defined in the positive region Λ . We establish conditions for existence of the equilibrium state and found both drug free and drug use endemic equilibrium point to be locally and asymptotically stable when $R_0 < 1$ and unstable otherwise. Numerical simulation results shows the increase in the drug user (both light and heavy) and the exposed population when there is no significant control applied to reduce the problem. The simulation further shows the significant decrease of the rate at which new drug users are influenced and thus reduce the number of drug users in the community which then increases the number of susceptible humans who in a way are not risky of becoming drug users.

Numerical results further indicate that the effective result of the control method depends on the effort applied. When the effort is high the chance to defeat the problem gets higher. The control method reduces the progression rate of the light drug users to heavy drug users. It is beyond doubt that the involvement of society is very vital to the successful control of drug abuse in society. This will help to get a holistic understanding of the nature and practice of the drug users and most people involved in the drug abuse business.

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Conflict of interest. The authors declare that they have no conflict of interest.

Authors' Contributions. All the authors contributed equally to this work.

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Output 2: Mathematical Modeling on the Effects of Drug Abuse to the Societies

MATHEMATICAL MODELING ON THE EFFECTS OF DRUG ABUSE TO THE SOCIETIES

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Introduction

Drug (s) stands for any chemical substance which when used without the directive of the physician affect living organism. These substance have got variety of use but mostly may be used for treatment of various health problems of a human being (Matowo,2013). On the other hand drugs are substances that produce various reactions or modify

Statement of the Problem

It is evident that some of the prominent scholars in the field of Drug Models have been formulated to study the effects of Alcohol in relation to HIV and TB but the role of the dynamics of the effects of drug abuse to the societies is considered less.

Table 2: Parameters' Description

Parameter	Description
λ	The rate of recruitment of human population
ψ_1 & ψ_2	The rate that S become E due to I_1 and I_2 respectively
α_1	Progression rate out of E to drug user state.
ρ	Proportional of E becoming I_2
σ_1	Rate at which individual leave I_1
ρ_1	Proportional of I_1 becoming I_2
ρ_2	Proportional of I_1 becoming Q
ρ_3	Proportional of I_1 becoming R
α_2	The rate at which Q become R
α_3	The rate at which R become S
μ_1	Natural death rate of human being.
μ_2	Drug addiction induced death rate of human being.
ω	Control efforts to protect S
N	Total human population.

Objectives

- To identify factors that affect drug abuse dynamics in the society.
- To formulate the basic model system for the dynamics of drug abuse to the society.
- To analyze the mathematical model for the effect of the drug abuse to the society.
- To explore the dynamical behavior, control(s) and effect of drug abuse in the society using numerical simulation .

Model development process

Compartmental Model Diagram

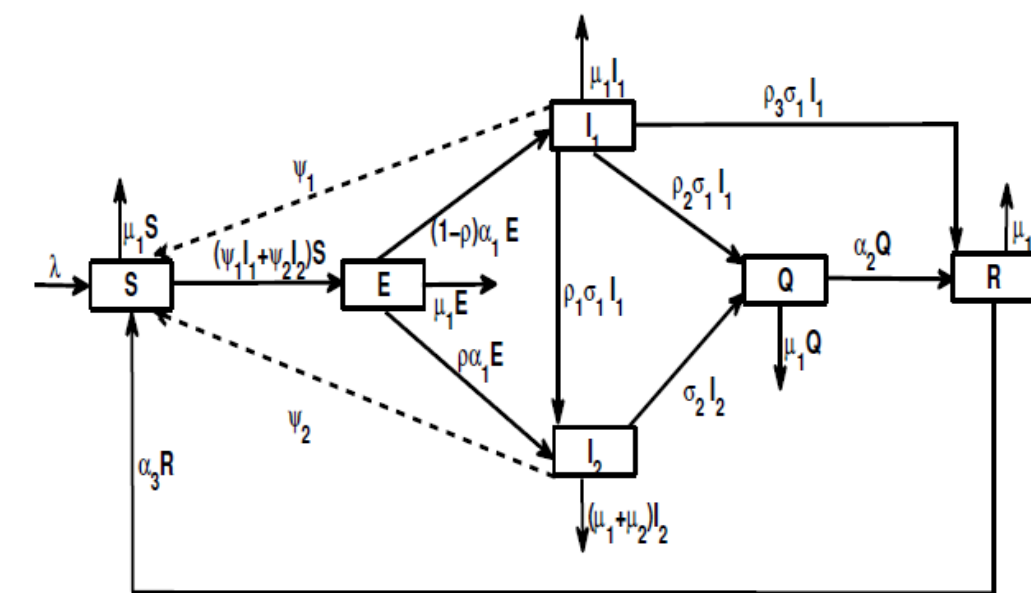


Figure 1: Compartmental model for Drug Use

Model Equations

$$\begin{aligned}
 \frac{dS}{dt} &= \lambda + \alpha_3 R - (\psi_1 I_1 + \psi_2 I_2)(1 - \omega)S - \mu_1 S \\
 \frac{dE}{dt} &= (\psi_1 I_1 + \psi_2 I_2)(1 - \omega)S - ((1 - \rho) + \rho)\alpha_1 E - \mu_1 E \\
 \frac{dI_1}{dt} &= (1 - \rho)\alpha_1 E - (\rho_1 \sigma_1 + \rho_2 \sigma_1 + \rho_3 \sigma_1 + \mu_1)I_1 \\
 \frac{dI_2}{dt} &= \rho\alpha_1 E + \rho_1 \sigma_1 I_1 - (\sigma_2 + \mu_1 + \mu_2)I_2 \\
 \frac{dQ}{dt} &= \rho_2 \sigma_1 I_1 + \sigma_2 I_2 - (\alpha_2 + \mu_1)Q \\
 \frac{dR}{dt} &= \rho_3 \sigma_1 I_1 + \alpha_2 Q - (\mu_1 + \alpha_3)R
 \end{aligned}
 \tag{1}$$

$\rho_1 + \rho_2 + \rho_3 = 1$

subject to initial conditions:

$$S > 0, E \geq 0, I_1 \geq 0, I_2 \geq 0, R \geq 0, Q \geq 0$$

Model analysis

❖ Disease Free Equilibrium is given by:

- The model has a drug free equilibrium which is obtained by setting all infectious compartment and the derivative equal to zero. Then the drug free-equilibrium point is as given.

$$X_0(S^0, E^0, I_1^0, I_2^0, Q^0, R^0) = \left(\frac{\lambda}{\mu_1}, 0, 0, 0, 0, 0 \right)$$

❖ The reproduction number R_0 is given by

$$R_0 = \frac{((\eta\psi_1 + \rho\psi_2)\mu_1 + \psi_2\rho_1(\eta + \rho)\sigma_1 + \eta\psi_1(\sigma_2 + \mu_2))\alpha_1\lambda}{\mu_1(\alpha_1 + \mu_1)(\sigma_2 + \mu_1 + \mu_2)(\rho_1\sigma_1 + \mu_1)}$$

Where $\eta = (1 - \rho)$.

.When early identification and quarantine control method is applied, the number of secondary cases of drug users will decrease which as a result will reduce the number of adequate contact between the drug user and the susceptible population. Now using the same next generation matrix method we can generate an effective reproduction number that consider the effect of control.

The disease-free equilibrium point E^0 is locally asymptotically stable if $R_0 < 1$ and it is unstable when $R_0 > 1$

Table 3: Parameter Values for the Model

Parameters	Value	Source
λ	5	Tang <i>et al.</i> (2021)
ψ_1	0.0001	Estimated
ψ_2	0.0003	Estimated
α_1	0.039	Glantz and Chambers (2006)
ρ	0.2	Njagarah and Nyabadza (2013)
σ_1	0.0301	Orwa and Nyabadza (2019)
ρ_1	0.4	Bhunu and Mushayabasa (2012)
μ_1	0.02	Bradshaw and Timaeus (2006)
ρ_2	0.3	Bhunu and Mushayabasa (2012)
ρ_3	0.3	Orwa and Nyabadza (2019)
α_2	0.0051	Bhunu and Mushayabasa (2012)
α_3	0.0089	Orwa and Nyabadza (2019)
μ_2	0.059	Oyefeso <i>et al.</i> (1999)

Numerical Simulation

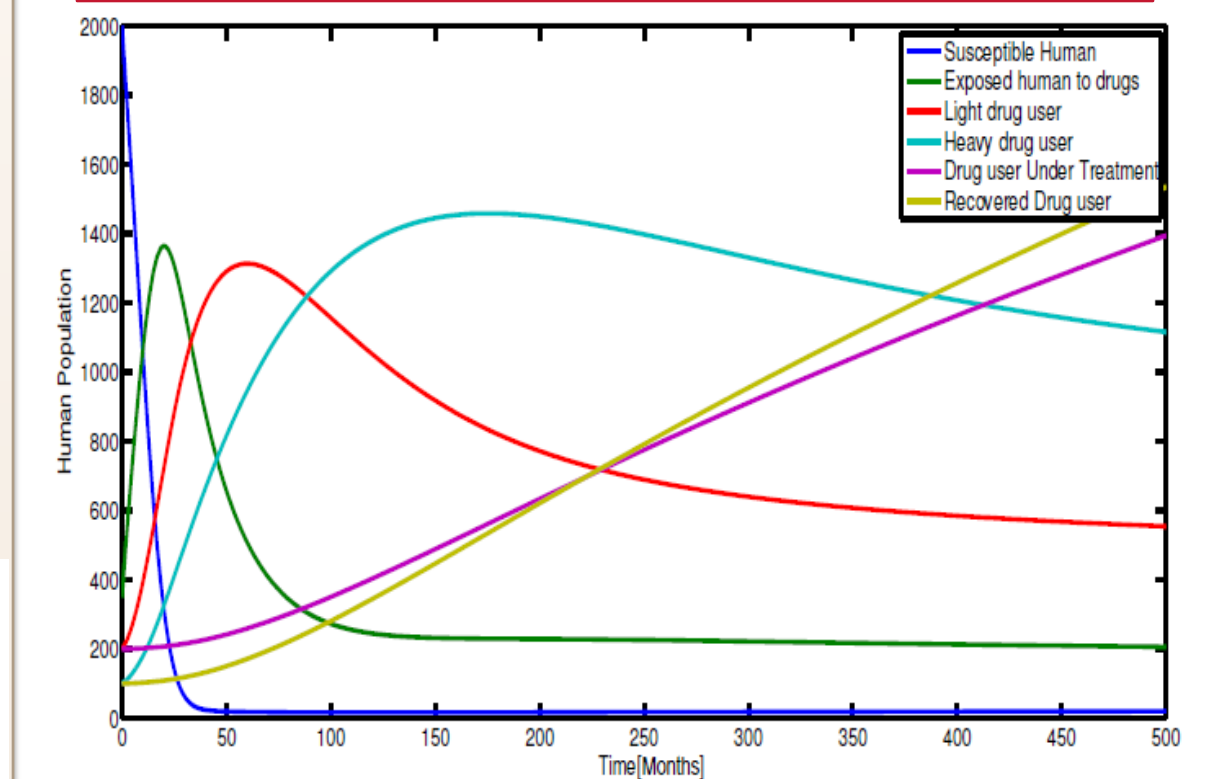


Figure 2: Behavioral Dynamics of Drug Abuse in Human Population

Conclusion

In this study, the mathematical model to study the dynamical behavior of the community with drug abuse is developed and analyzed, using numerical simulation results shows the dynamical behavior of the population in their classes in the community with drug abuse. The result shows the increase in the drug user (both light and heavy) and the exposed population when there is no significant control applied to reduce the problem.

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